(ALMOST) EVERYTHING YOU NEED TO REMEMBER ABOUT TRIGONOMETRY, IN ONE SIMPLE DIAGRAM

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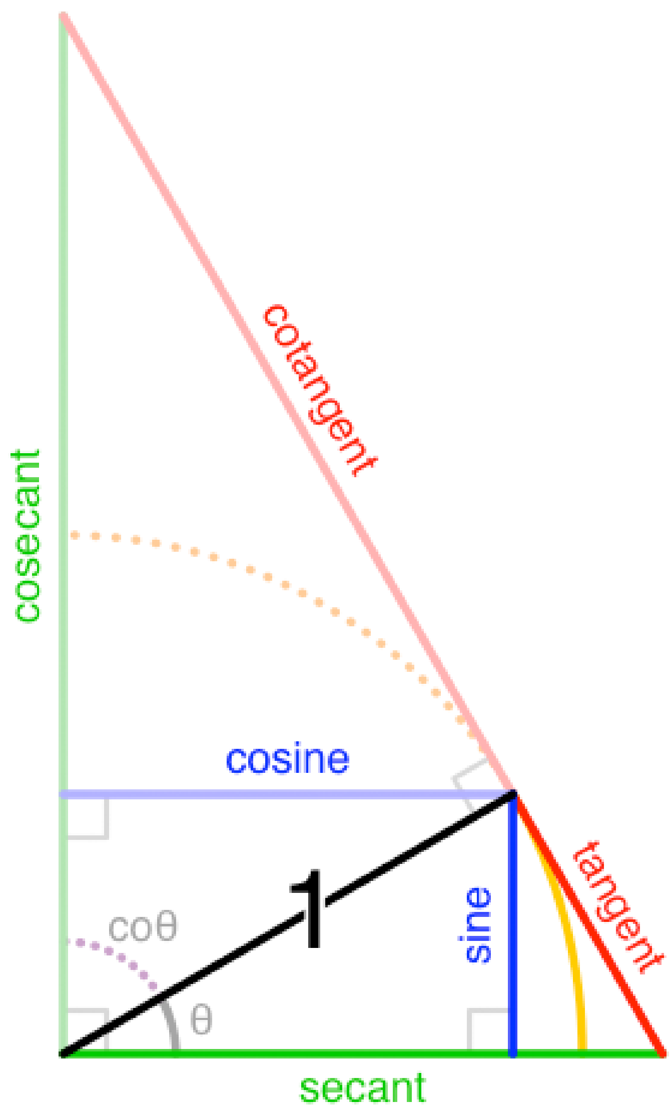

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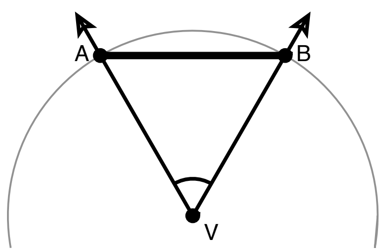
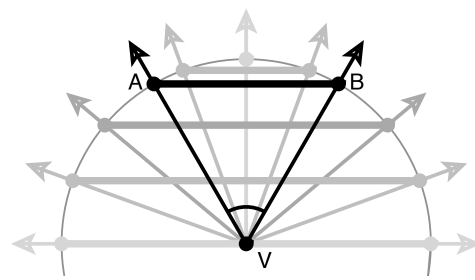
1. Introduction

Trigonometry —literally, "triangle measurement"— is the study of the interplay between two wildly different notions of measurement: the *lengths of segments* and the *sizes of angles*.

As we all know, an angle is defined by a pair of rays emanating from a common point called the vertex (V). If we were to pick two points (A and B), one on each ray, the same distance from the vertex, our Euclidean instincts might compel us to draw the line joining them; here, we'll make do with a segment. We might also draw a circle with center V passing through A and B, recognizing the segment as a *chord* of that circle.



(a) A chord spanning an angle.



(b) The chord changes as the angle changes.

Figure 1. A chord associated with an angle.

The connection between the chord and the angle —between the *length* of the chord and the *size* of the angle— is clearly fairly tight. (Provided that we're always talking about A and B being the same chosen distance from V,) An angle with the smallest possible size ($0°$) would yield a chord with the shortest possible length (0); an angle of the largest possible size[1] ($180°$) would yield the longest possible chord (with length equal to the diameter of the circle, which is twice the chosen distance from A or B to V); somewhere in there is a chord tied to the right angle ($90°$), or to any angle you prefer.

Note that, while the *qualitative* connection is clear —the wider the angle, the longer the chord— the *quantitative* connection is extremely murky. The chord that "goes with" the $180°$ angle is not twice as long as the chord that goes with the $90°$ angle; rather, it's twice as long as the chord that goes with the $60°$ angle! Part of understanding trigonometry is coming to grips with the seemingly bizarre nature of that connection,[2] but, for the moment, the point is that the connection *exists*: If you know the size of the angle, then you can (somehow) infer the length of the chord, and vice-versa. The connection, murky though it may be, makes the information virtually interchangeable, which gives us problem-solving options.

That said, trigonometric tradition features not one, but *six*[3] segments that "go with" an angle; as it turns out, none of the segments is the chord described above (although they all relate to it). Each trig segment its own special connection to the angle and offers its own special problem-solving advantage (not to mention: disadvantages). These segments —more precisely, their lengths— give rise to what we know as the standard trig functions and their curious names: sine, tangent, secant,

[1] For the purposes of this discussion, angles don't get bigger than a straight angle.

[2] And that's *not easy!* Although the connection had been studied and used for thousands of years, the *formula* for converting angle sizes into chord lengths wasn't known until Madhava at the turn of the 1400s in the East or Isaac Newton in the 1670s in the West. (How do I cite the on-line *Encyclopedia Britannica*?) Before then, the state of the art was effectively to compile look-up tables of values, a practice that hand-held calculators made obsolete only a few decades ago.

[3] Actually, there were more (e.g., "versine" and "exsecant"), but they are no longer common use.

cosine, cotangent, and cosecant. Sadly, many (if not most) modern treatments of trigonometry introduce the trig functions without regard to the underlying segments, robbing those functions of their geometry and their *meaning*, and, thus, making the subject appear arbitrary and cryptic to students. This note seeks to restore the segments to prominence, combining them into a remarkable figure called the *Fundamental Trigonograph*[4] that promotes *comprehension* (not merely *memorization*) of many fundamental principles of trigonometry.

2. The Fundamental Trigonograph

We'll adopt a conventional setting by starting with the unit circle in the coordinate plane. That is, consider a circle of radius 1, centered at the origin (O). For the time being, we'll restrict our attention to the First Quadrant, letting X and Y be the points where the circle crosses the x- and y-axes, and choosing P to be a point somewhere in the quarter-circular arc between them. The angle $\angle XOP$, which we'll say has measure θ, is the focus of our attention.

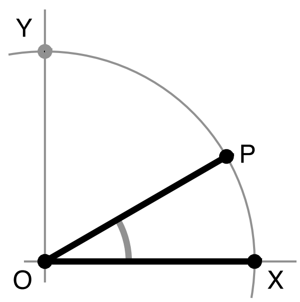


Figure 2. A First Quadrant angle.

The first of trig's segments that "go with" θ is the perpendicular dropped from P to the x-axis. Its seemingly unmotivated construction traces back to the segment of the Introduction: if we mirror $\angle POX$ in the x-axis, then we see this segment as *half* of the chord spanning *twice* the angle θ. Over the years, this *semi-chord* stole the mathematical spotlight from the full chord, and has come to be known as the "sine segment". (See Section 3.)

The second trig segment is the perpendicular "raised" from P to the x-axis. Being at right angles to the circle's radius $\overline{OP}$, this segment is tangent to the circle, earning the name of *tangent* segment.[5]

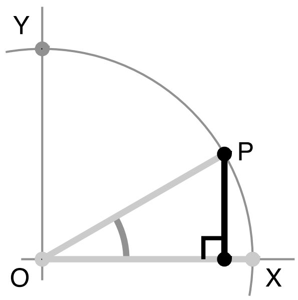
The third trig segment has already been determined: It joins the origin to the point where the tangent segment meets the x-axis. As part of a line passing through the circle, it receives the name *secant* segment.[6]

The lengths of these segments are as intimately connected to the size of the angle as the chord discussed in the Introduction. Again, the exact formulaic connection is tricky (and beside the point here), but the general sense is clear: bigger angles have longer segments, and to know the angle's

[4]"Trigonograph" is this author's name for a geometric diagram that illustrates a trigonometric relation or concept, ideally in a way that makes things "obvious". The term is a back-construction of the word-playful *trigonography*, which could be taken to mean "the art of trigonometric visualization".

[5]Alternatively, we could have defined this segment by raising a perpendicular from X to an extension of radius $\overline{OP}$. Doing this, and mirroring $\angle POX$ as we did with sine, would reveal the segment to be *half* the tangent segment spanning *twice* the angle θ in a natural way.

[6]If we think of this segment as being *half* of some other (an insight useful for sine and tangent), then that other would be a proper secant segment, passing all the way through the circle.

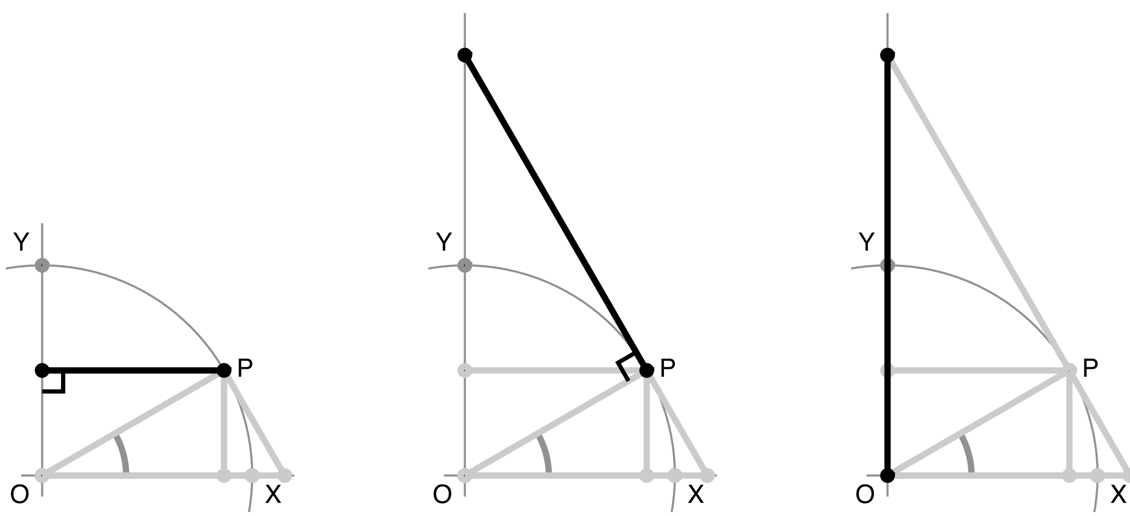


(a) *sine* (a semi-chord). (b) *tangent* (a tangent). (c) *secant* (a secant).

FIGURE 3. Three segments associated with $\angle XOP$.

size is to know each segment's length (and vice-versa). We'll cover specific observations about these connections in later sections of this note.

The final three trig segments form (*ahem*) a *perfect complement* to the first three, because they're constructed in the same way, but relative to $\angle POY$, which is the *complement* of $\angle POX$: the perpendicular dropped from P to the y-axis is the *complementary sine* —that is, the *co-sine*— segment; the perpendicular *raised* from P to the y-axis is the *complementary tangent* (*co-tangent*) segment; and the *complementary secant* (*co-secant*) segment joins origin to the intersection of the co-tangent segment and the y-axis.



(a) *cosine* (a semi-chord). (b) *cotangent* (a tangent). (c) *cosecant* (a secant).

FIGURE 4. Three co-segments associated with $\angle XOP$. (Equivalently, the three "ordinary" segments associated with $\angle XOP$'s *complement*, $\angle YOP$).

Together, these six segments, along with the radius segment $\overline{OP}$ of length 1, form the figure we call the "Fundamental Trigonograph". As we'll see, the geometry of the segments neatly encodes numerous properties of, and relationships among, the corresponding trigonometric functions.

Until further notice, we restrict our attention to non-obtuse angles.

3. Etymological-Definitional Properties

The names of the six trig segments directly reflect their geometric origins, as described in the previous section. In particular, each name expresses a relation to the defining unit circle in the Fundamental Trigonograph:

The	*sine*	and	*cosine*	segments are	*semi-chords*	of the unit circle.[7]
The	*tangent*	and	*cotangent*	segments are	*tangents*	of the unit circle.
The	*secant*	and	*cosecant*	segments are	*secants*	of the unit circle.

Moreover, the "co" prefix (short for "complementary") connects three of the segments to the complement of the trigonograph's defining angle. Abusing that prefix, we use "coθ" to mean "the complement of θ", so that we can write:

The	*cosine*	segment for θ is the	*sine*	segment for coθ.	$\cos(\theta) = \sin(\text{co}\theta)$
The	*cotangant*	segment for θ is the	*tangent*	segment for coθ.	$\cot(\theta) = \tan(\text{co}\theta)$
The	*cosecant*	segment for θ is the	*secant*	segment for coθ.	$\csc(\theta) = \sec(\text{co}\theta)$

The trigonograph's radius segment visually separates what we might call "ordinary elements" (angle θ, and segments sine, tangent, secant) from "complementary elements" (coθ, cosine, cotangent, cosecant); "complementing", then, is merely a matter of jumping over the radius. Naturally, jumping *twice* is like not jumping at all, a maxim reflected in the fact that *the complement of an angle's complement is the angle itself* ("cococoθ" is θ), as well as these relations:

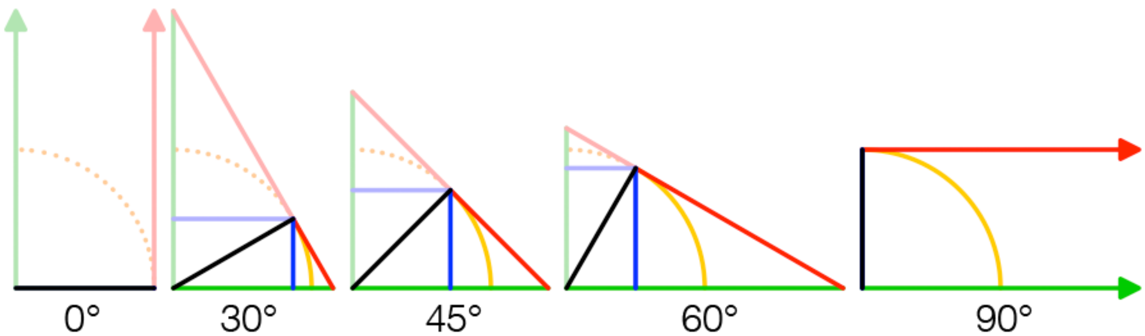
The	*cosine*	segment for coθ is the	*sine*	segment for θ.	$\cos(\text{co}\theta) = \sin(\theta)$
The	*cotangant*	segment for coθ is the	*tangent*	segment for θ.	$\cot(\text{co}\theta) = \tan(\theta)$
The	*cosecant*	segment for coθ is the	*secant*	segment for θ.	$\csc(\text{co}\theta) = \sec(\theta)$

Thus, when considered in their native geometrical context, the six trig values have names that explain not only *what they are*, but also the very basics of *how they relate*. As we'll see, that geometrical context provides a wealth of additional information.

[7] We can blame medieval scholars for lack of obvious-ness here. The originally-obvious Sanskrit name *ardha-jiva* ("half-chord") shortened over time to *jiva*, which became the Arabic *jiba*, which was confused for *jaib* ("bay") when translated into Latin as *sinus* ("bay" or "fold"). See, for instance, "History of trigonometric functions" on Wikipedia. `http://en.wikipedia.org/w/index.php?title=History\_of\_Trigonometric\_functions\&oldid=85828396`

4. Dynamic Properties

The Fundamental Trigonograph can be viewed as a *dynamic* figure, with its radius segment actively *sweeping* through the entire quarter circle, creating every angle from the zero angle to the right angle. All the while, the six trig segments grow and shrink in concert, exhibiting certain patterns.



4.1. Fun-House Mirror Properties.
The trigonograph's radius segment —which is neither ordinary, nor complementary (or is it *both*?)— acts something like a fun-house mirror between the two groups: through it, each trig element can see its co-element as a distorted version of itself, the two of them growing or shrinking in opposing manners.

As	θ	gets larger (smaller),	coθ	gets smaller (larger).
As	*sine*	gets larger (smaller),	*cosine*	gets smaller (larger).
As	*tangent*	gets larger (smaller),	*cotangent*	gets smaller (larger).
As	*secant*	gets larger (smaller),	*cosecant*	gets smaller (larger).

Importantly, these observations are *qualitative* ("larger" vs "smaller"), not *quantitative* ("*how much* larger" vs "*how much* smaller").

Not only do the ordinaries simultaneously *get* larger or smaller than complementaries, at any given time, they *are*:

The Herd Mentality Property
All ordinary elements are either simultaneously larger than, simultaneously equal to, or simultaneously smaller than, their respective co-ordinary elements.

In particular, co-elements *match* at the trigonograph's "half-way" state, when θ and coθ are 45°.

4.2. Range Properties.
The Fundamental Trigonograph's "extreme" states —when θ is a zero angle or a right angle— feature trig segments at their smallest or largest. In these states, some trig segments have collapsed into length-less points, others have coincided with the unit-length radius segment, and yet others have extended into infinitely-long *rays*. The *ranges* of the corresponding trig values are as follows:

	$\theta = 0°$	$\theta = 90°$	range
sine	point	radius	0 to 1
tangent	point	ray	0 to ∞
secant	radius	ray	1 to ∞
cosine	radius	point	1 to 0
cotangent	ray	point	∞ to 0
cosecant	ray	radius	∞ to 1

TABLE 1. Extreme states of the six trig segments.

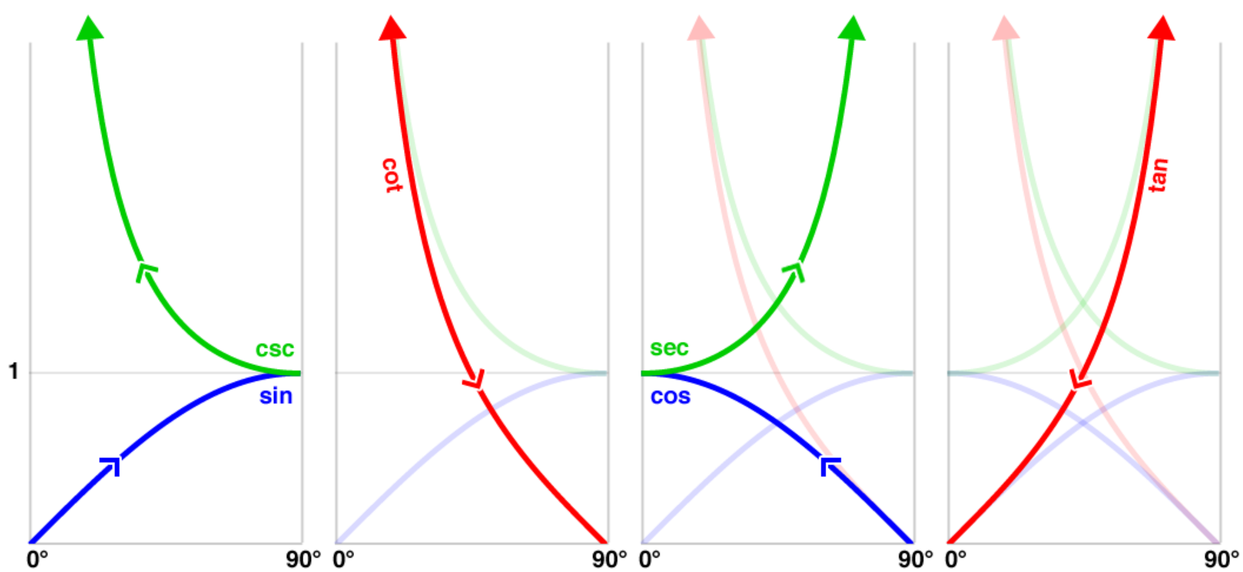
As one should expect, the range of each element is the reverse of its co-element. Moreover, in accordance with the fun-house mirror properties, as θ sweeps through its own range of 0° to 90°, ordinary elements *strictly increase* in their range, while complementary elements *strictly decrease*.

The Fundamental Trigonograph's "half-way" state is a half-square, with the radius, tangent, and cotangent segments appearing as congruent half-diagonals. Thus, at the beginning, middle, and end of θ's range, *some* pair of trig segments attain the length 1.

Interestingly, the ranges can be placed end-to-end in a chain, thusly:

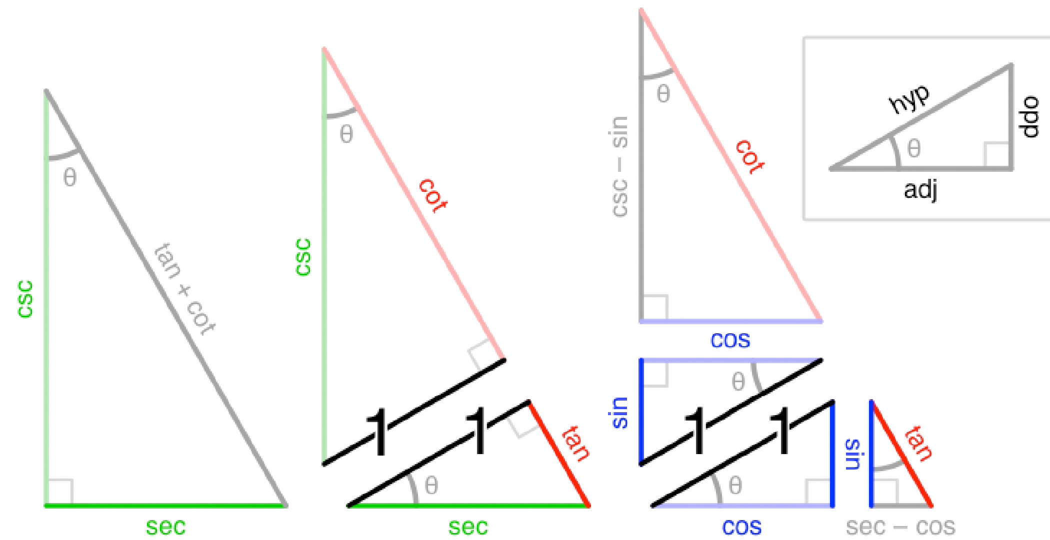
0	sin	1	csc	∞	cot	0	cos	1	sec	∞	tan	0
0°		90°		0°		90°		0°		90°		0°

which leads to a convenient way to sketch the *graphs* of all six trig values, without lifting one's pen:



5. Static Properties (Identities and Comparaties)

Perhaps the most-important feature of the Fundamental Trigonograph is that it contain *six similar right triangles* (plus a duplicate), each having at least two trig segments as sides.



Each triangle contains the angle θ (and thus also coθ, not marked in the figure). We describe a leg of the triangle as being either *opposite* ("opp") or *adjacent* ("adj") to angle θ, as shown in the generic "opp-adj-hyp" triangle above (where "hyp", of course, refers to the *hypotenuse*). We name each triangle by the pair of trig segments it contains, as follows:

		hypotenuse	opposite leg	adjacent leg
⋆	sin-cos	1	sin	cos
⋆	tan-sec	sec	tan	1
⋆	cot-csc	csc	1	cot
	sin-tan	tan	sec − cos	sin
	cos-cot	cot	cos	csc − sin
	sec-csc	tan + cot	sec	csc

Table 2. The six triangles of the Fundamental Trigonograph.

The table groups each triangle with its complementary counterpart (sin-cos and sec-csc are their own co-unterparts). The table also marks with "⋆" those triangles that have the radius segment (that is, 1) as a side, as these have greatest significance to our work.

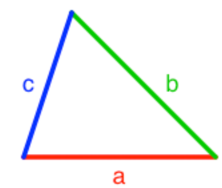
Leveraging geometric theorems regarding similar right triangles, we can derive a number of key relations that hold in whatever state the Fundamental Trigonograph assumes. That is, the relations hold *regardless of the angle involved.* When such a universal relation is an *equality* it is called an "identity" (because the equated elements are considered *logically identical*, expressing the same value in different ways). So far as this author knows, there's no official term for a universal *comparison*, so we'll use the invented word "comparaty". Let's discuss those first.

5.1. **Comparaties.** The *Triangle Inequality* is geometry's most-basic expression that *the shortest distance between two points is a straight line*, stating that it's quicker to go from one vertex of a triangle to another by their joining edge, rather than "the long way around" via the third vertex.

The Triangle Inequality

The sum of the lengths of two sides of a triangle is greater than (or equal to[8]*) the length of the third side.*

$$a + b \geq c \qquad b + c \geq a \qquad c + a \geq b$$



For our triangles, this says that $adj + hyp \geq opp$, $opp + hyp \geq adj$, and $opp + adj \geq hyp$. But our triangles are *right* triangles, subject to a stricter rule:

The Hypotenuse Inequality

The hypotenuse is the longest side of a right triangle.

This allows us to strengthen the first two Triangle Inequality statements, so that we have

$$hyp \geq opp \qquad hyp \geq adj \qquad opp + adj \geq hyp$$

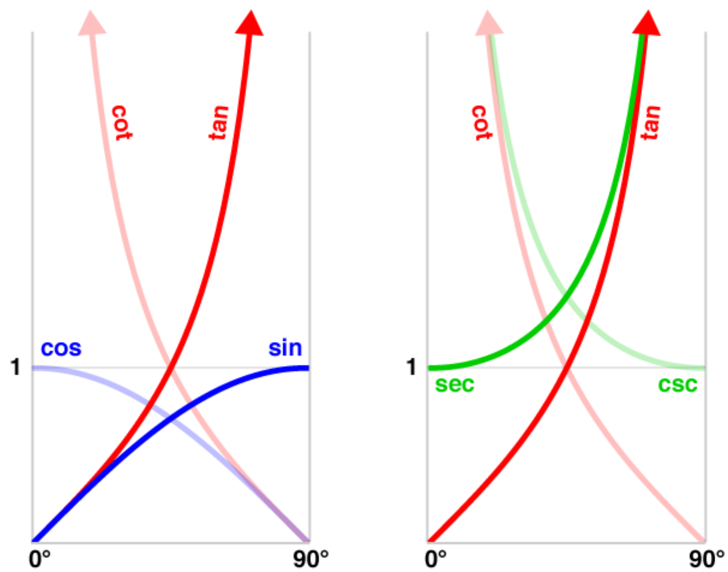
Table 3 lists the specific comparaties that arise from applying these facts to our triangles.

					0°	90°
⋆	sin-cos	1	$\geq$	sin	·	$1 = 1$
⋆	sin-cos	1	$\geq$	cos	$1 = 1$	·
⋆	tan-sec	sec	$\geq$	1	$1 = 1$	·
⋆	cot-csc	csc	$\geq$	1	·	$1 = 1$
∘	tan-sec	sec	$\geq$	tan	·	"$\infty = \infty$"
∘	cot-csc	csc	$\geq$	cot	"$\infty = \infty$"	·
∘	sin-tan	tan	$\geq$	sin	$0 = 0$	·
∘	cos-cot	cot	$\geq$	cos	·	$0 = 0$
	tan-sec	$\tan + 1$	$\geq$	sec	$1 = 1$	"$\infty = \infty$"
	cot-csc	$\cot + 1$	$\geq$	csc	"$\infty = \infty$"	$1 = 1$
	sin-cos	$\sin + \cos$	$\geq$	1	$1 = 1$	$1 = 1$
	sin-tan	tan	$\geq$	$\sec - \cos$	$0 = 0$	"$\infty = \infty$"
	cos-cot	cot	$\geq$	$\csc - \sin$	"$\infty = \infty$"	$0 = 0$
	sin-tan	$\sec - \cos + \sin$	$\geq$	tan	$0 = 0$	"$\infty = \infty$"
	cos-cot	$\csc - \sin + \cos$	$\geq$	cot	"$\infty = \infty$"	$0 = 0$
	sec-csc	$\tan + \cot$	$\geq$	sec	"$\infty = \infty$"	"$\infty = \infty$"
	sec-csc	$\cot + \tan$	$\geq$	csc	"$\infty = \infty$"	"$\infty = \infty$"
	sec-csc	$\sec + \csc$	$\geq$	$\tan + \cot$	"$\infty = \infty$"	"$\infty = \infty$"

TABLE 3. Eighteen comparaties, and grouped by significance and in complementary pairs. In each case, if equality occurs at all, it occurs at 0° or 90°.

[8] The "or equal to" aspect applies to so-called *degenerate* (i.e., "flat") triangles for which that third vertex lies between the other two, making "the long way around" the same as the direct route. This note embraces such degeneracy, while other sources do not; the reader is encouraged to regard neither approach as absolutely (in)correct.

The comparaties marked "$\star$" reconfirm some previously-described range properties. Those marked "$\circ$" are clearly visible in the graphs of the trigonometric functions:



5.2. **Pythagorean Identities.** The most well-known result in all of geometry may be this:

> **The Theorem of Pythagoras**
> *The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.*
>
> $$opp^2 + adj^2 = hyp^2$$

Table 4 applies this theorem to the Fundamental Trigonograph's six triangles. Note that each identity holds if each element is replaced with its co-element.

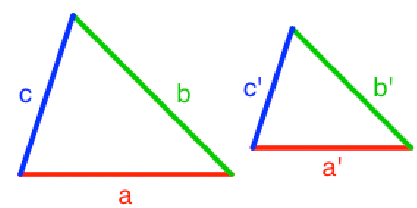
$\star$	sin-cos	$\sin^2 + \cos^2$	$=$	1
$\star$	tan-sec	$\tan^2 + 1$	$=$	$\sec^2$
$\star$	cot-csc	$\cot^2 + 1$	$=$	$\csc^2$
	sin-tan	$\sin^2 + (\sec - \cos)^2$	$=$	$\tan^2$
	cos-cot	$\cos^2 + (\csc - \sin)^2$	$=$	$\cot^2$
	sec-csc	$\sec^2 + \csc^2$	$=$	$(\tan + \cot)^2$

TABLE 4. Pythagorean identities

The items marked "$\star$" are precisely those involving 1; these three are typically called "*the* Pythagorean identities", and the first of them is the most widely known. The others are interesting curiosities.

5.3. **Internal Proportionality Properties.** A family of similar triangles admits the following proportionality property, called "internal" because each ratio uses components of the same triangle:

> **The Internal Proportion Theorem**
> *The ratio of lengths in a triangle is equal to the ratio of corresponding lengths in any similar triangle.*
>
> $$\frac{a}{b} = \frac{a'}{b'} \qquad \frac{b}{c} = \frac{b'}{c'} \qquad \frac{c}{a} = \frac{c'}{a'}$$



Right triangles with a common angle θ are necessarily similar, and their corresponding sides are conveniently identified by common descriptors "opp", "adj", "hyp". Therefore, we can interpret the above thusly:

> *Each of the following is constant across all right triangles in the Fundamental Trigonograph:*
>
> $$\frac{opp}{hyp} \qquad \frac{adj}{hyp} \qquad \frac{opp}{adj}$$

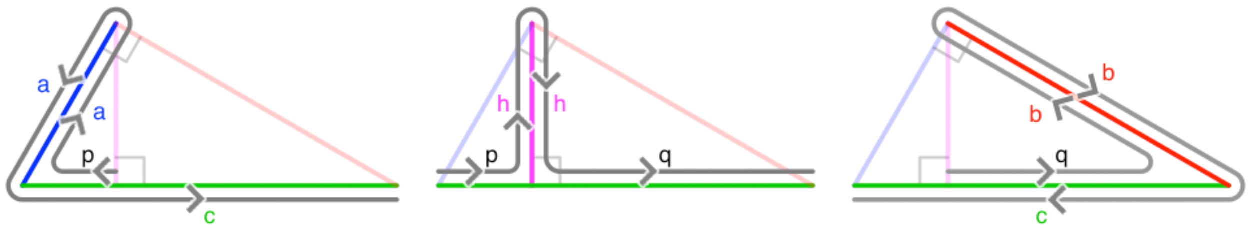
Table 5 applies this rule to our six triangles, logging fully forty-five individual identities (or ninety, if you count reciprocation). Note that any particular identity remains true if all elements are swapped with their co-elements.

	sin-cos		tan-sec		cot-csc		sin-tan		cos-cot		sec-csc
$\frac{\text{opp}}{\text{hyp}}$	$\frac{\sin}{1}$	$=$	$\frac{\tan}{\sec}$	$=$	$\frac{1}{\csc}$	$=$	$\frac{\sec - \cos}{\tan}$	$=$	$\frac{\cos}{\cot}$	$=$	$\frac{\sec}{\tan + \cot}$
$\frac{\text{adj}}{\text{hyp}}$	$\frac{\cos}{1}$	$=$	$\frac{1}{\sec}$	$=$	$\frac{\cot}{\csc}$	$=$	$\frac{\sin}{\tan}$	$=$	$\frac{\csc - \sin}{\cot}$	$=$	$\frac{\csc}{\tan + \cot}$
$\frac{\text{opp}}{\text{adj}}$	$\frac{\sin}{\cos}$	$=$	$\frac{\tan}{1}$	$=$	$\frac{1}{\cot}$	$=$	$\frac{\sec - \cos}{\sin}$	$=$	$\frac{\cos}{\csc - \sin}$	$=$	$\frac{\sec}{\csc}$

TABLE 5. Internal proportionality identities.

5.3.1. *Geometric Mean Properties.* The *geometric mean* of numbers m and n is the number g such that $m/g = g/n$; that is, $mn = g^2$, so that $g = \sqrt{mn}$.[9] The extensive lore of such means is beyond the scope of the current discussion, but it's worth mentioning how easily they arise in right triangles.

[9]Thus, the geometric mean of two numbers is their *product*, raised to the *power* of 1/2. Compare this to the numbers' *arithmetic mean*, which is their *sum*, multiplied by a *factor* of 1/2.



> *Dropping a perpendicular to the hypotenuse of a right triangle creates three geometric mean configurations whose components you can trace with your finger.*
>
> $\frac{p}{a} = \frac{a}{c} \;\to\; pc = a^2 \qquad \frac{p}{h} = \frac{h}{q} \;\to\; pq = h^2 \qquad \frac{q}{b} = \frac{b}{c} \;\to\; cq = b^2$

The reason the geometric means appear is because the added perpendicular sub-divides a right triangle into similar sub-triangles, whereupon the Internal Proportion Theorem relates the lengths. Thus, insofar far as this result applies to the Fundamental Trigonograph, we don't see anything in Table 6 that we hadn't seen in Table 5; we're only seeing some of those things in a new way.

	tan-sec	cot-csc	sec-csc
$\star$	$\cos \cdot \sec = 1^2$	$\sin \cdot \csc = 1^2$	$\tan \cdot \cot = 1^2$
	$\cos \cdot (\sec - \cos) = \sin^2$	$\sin \cdot (\csc - \sin) = \cos^2$	$\tan \cdot (\tan + \cot) = \sec^2$
	$(\sec - \cos) \cdot \sec = \tan^2$	$(\csc - \sin) \cdot \csc = \cot^2$	$\cot \cdot (\cot + \tan) = \csc^2$

TABLE 6. Geometric mean identities.

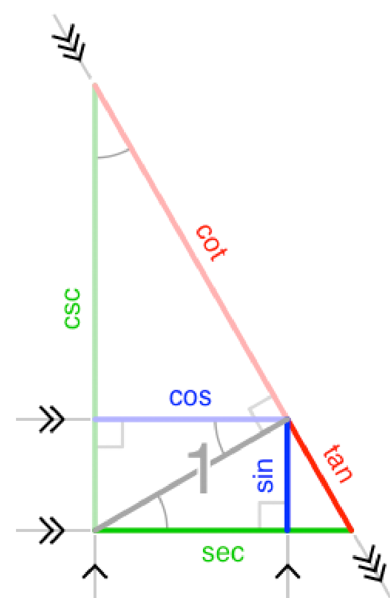
The identities marked "$\star$" express key reciprocal relations that can be a little off-putting at first glance. Why isn't *CO-secant* the reciprocal of *CO-sine*? Why are *tangent* and *CO-tangent* reciprocals, but not *sine* and *CO-sine* (or *secant* and *CO-secant*)? It all seems so frustratingly arbitrary.

To ease some of that frustration, simply remember that "co" means "complementary"; nothing more, nothing less. Complementary elements are related to ordinary elements by *complementary angles*, and one should not *expect* anything more of them or their reciprocals. (That co-elements tangent and co-tangent are reciprocals is merely a co-incidence.)

Also, keep in mind that the relations are *right there in the Fundamental Trigonograph.* (That's the whole point of this document!) If you forget them, just find the appropriate sub-triangles and trace the appropriate geometric means.

If all that's too much trouble, then here's a handy tip:

> *In the Fundamental Trigonograph, reciprocal segments are parallel.*
>
> $\sin \parallel \csc \qquad \cos \parallel \sec \qquad \tan \parallel \cot$



5.3.2. *"Textbook Definition" Properties.* Amid the myriad Internal Proportionality Identities are these relations:

"Textbook Definitions" of tan**,** sec**,** csc**,** cot

$$\frac{\tan}{1} = \frac{\sin}{\cos} \qquad \frac{\sec}{1} = \frac{1}{\cos} \qquad \frac{\csc}{1} = \frac{1}{\sin} \qquad \frac{\cot}{1} = \frac{1}{\tan}\left(= \frac{\cos}{\sin}\right)$$

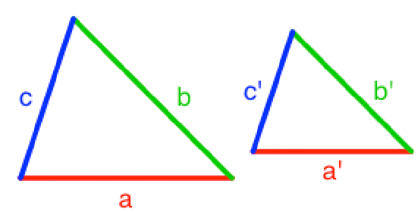
Some (most? all?) modern trig textbooks treat these as *definitions* —"*tangent* is *sine-over-cosine*", "*secant* is *1-over-cosine*", etc.— only later (if ever) explaining the underlying geometry. Needless to say, this author prefers the geometric approach.

5.4. **External Proportionality Properties.** A pair of similar triangles exhibit this proportionality property, called "external" because each ratio involves components from separate triangles:

The External Proportion Theorem

The ratios of the sides a, b, c of a triangle to respective sides a', b', c' of any similar triangle are equal.

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$



Therefore:

For any two right triangles in the Fundamental Trigonograph,

$$\frac{opp}{opp'} = \frac{adj}{adj'} = \frac{hyp}{hyp'}$$

All told, the forty-five (or ninety) individual identities shown below in Table 7 duplicate those derived from the internal proportions in Table 5. Even so, it's worth seeing those identities in the context of external proportions.

	tan-sec	sin-tan	sec-csc
sin-cos	$\frac{\sin}{\tan} = \frac{\cos}{1} = \frac{1}{\sec}$	$\frac{\sin}{\sec - \cos} = \frac{\cos}{\sin} = \frac{1}{\tan}$	$\frac{\sin}{\sec} = \frac{\cos}{\csc} = \frac{1}{\tan + \cot}$
tan-sec	$\cdot$	$\frac{\tan}{\sec - \cos} = \frac{1}{\sin} = \frac{\sec}{\tan}$	$\frac{\tan}{\sec} = \frac{1}{\csc} = \frac{\sec}{\tan + \cot}$
cot-csc	$\cdot$	$\frac{1}{\sec - \cos} = \frac{\cot}{\sin} = \frac{\csc}{\tan}$	$\frac{1}{\sec} = \frac{\cot}{\csc} = \frac{\csc}{\tan + \cot}$
sin-tan	$\cdot$	$\cdot$	$\frac{\sec - \cos}{\sec} = \frac{\sin}{\csc} = \frac{\tan}{\tan + \cot}$
cos-cot	$\cdot$	$\cdot$	$\frac{\cos}{\sec} = \frac{\csc - \sin}{\csc} = \frac{\cot}{\tan + \cot}$

	cot-csc	cos-cot	
sin-cos	$\frac{\sin}{1} = \frac{\cos}{\cot} = \frac{1}{\csc}$	$\frac{\sin}{\cos} = \frac{\cos}{\csc - \sin} = \frac{1}{\cot}$	$\uparrow$
tan-sec	$\frac{\tan}{1} = \frac{1}{\cot} = \frac{\sec}{\csc}$	$\frac{\tan}{\cos} = \frac{1}{\csc - \sin} = \frac{\sec}{\cot}$	$\uparrow$
cot-csc	$\cdot$	$\frac{1}{\cos} = \frac{\cot}{\csc - \sin} = \frac{\csc}{\cot}$	$\uparrow$
sin-tan	$\cdot$	$\frac{\sec - \cos}{\cos} = \frac{\sin}{\csc - \sin} = \frac{\tan}{\cot}$	$\uparrow$
cos-cot	$\cdot$	$\cdot$	$\uparrow$

Table 7. External proportionality identities.

5.5. **Area Properties.** We can derive a few additional identities by observing that the area of a right triangle can be computed in two ways:

$$\boxed{\begin{array}{ccccc} 2 \cdot \text{area} & = & a \cdot b & = & c \cdot d \\ & & (\text{ leg } \cdot \text{ leg }) & & (\text{ hyp } \cdot \text{ altitude-to-hyp }) \end{array}}$$

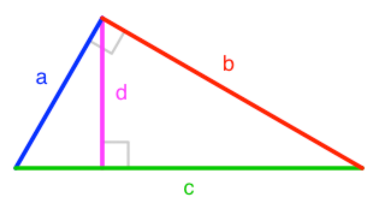


Table 8 applies this result to the largest three triangles of the Fundamental Trigonograph. As always, each identity holds if each element is replaced by its co-element.

	leg $\cdot$ leg		hyp $\cdot$ alt
tan-sec	$\tan \cdot 1$	$=$	$\sec \cdot \sin$
cot-csc	$\cot \cdot 1$	$=$	$\csc \cdot \cos$
sec-csc	$\sec \cdot \csc$	$=$	$(\tan + \cot) \cdot 1$

Table 8. Area identities.

6. Beyond the Right Angle

The Fundamental Trigonograph comprises elements related to an *acute* angle (or, at its extremes, the *right* angle or the *zero* angle). Even so, we can expand our understanding of the six trig values to accommodate angles of *any* size (and *either* sign!).[10]

To help motivate this discussion, let's take a look at a couple of quick trigonometric formulas. Here's a quick outline.

- The structure of the Fundamental Trigonograph is built off of an angle's "reference angle" in a quadrant.
  - Sine is a vertical semi-chord, and cosecant is (part of) a vertical secant line.
  - Cosine is a horizontal semi-chord, and cosecant is (part of) a horizontal secant line.
  - Tangent and cotangent are (parts of) the line tangent to the endpoint of the radius segment(with tangent connecting to the secant segment and cotangent connecting to the cosecant segment).
- The values of the trig functions are *signed lengths* of the corresponding segments, according to some seemingly-arbitrary rules (which I'll justify below).
  - Sine and cosecant are *positive* when these segments fall above the x-axis (Quadrants I and II), and *negative* when the segments fall below the x-axis (Quadrants III and IV).
  - Cosine and secant are *positive* when the segments fall to the right of the y-axis (Quadrants I and IV), and *negative* when the segments fall to the left of the y-axis (Quadrants II and III).
  - Tangent and cotangent are *positive* in Quadrants I and III, and *negative* in Quadrants II and IV. A good exercise for students is devise a Quadrant-free counterpart for this rule. Here's a decent attempt: Tangent is *positive* when the segment "points clockwise" from the point of tangency, and *negative* when the segment "points counter-clockwise". Since cotangent always points in the opposite direction to tangent, but always share's tangent's sign, the rule isn't so much that "clockwise = positive" and "counterclockwise = negative"; it's more that "tangent *prefers* clockwise" and "cotangent *prefers* counter-clockwise" ("co-clockwise"?).
- The sign assignments make secant and tangent "undefined" at angles coterminal with 90° and 270°; they make cosecant and cotangent "undefined" at angles coterminal with 0° and 180°. To see why, consider the Fundamental Trigonographs associated with angles *extremely close* to either side of $\theta = 90$. The tangent segment for θ ever-so-slightly-less-than 90° is an extremely large *positive* number; the closer θ gets to 90°, the more extreme the largeness, until $\theta = 90°$ is expected to give a tangent value of ∞. On the other hand, the tangent segment for θ ever-so-slightly-more-than 90° is an extremely large *negative* number; the closer θ gets to 90° in that case, the more extreme the largeness, until $\theta = 90°$ is expected to give a tangent value of $-\infty$. Because both ∞ and $-\infty$ are equally justifiable, the expression "$\tan 90°$" is (in this larger context) ambiguous, making the value "undefined".

6.1. **Explaining the Signs.** I don't normally introduce the Unit Circle as the vehicle for defining sine and cosine for arbitrary angles. I prefer to build toward a "full circle" (and beyond) view of Trigonometry by establishing a firm foundation of understanding in Quadrant I alone, focussing as much as possible on geometric principles. For one thing, this helps keep Geometry from being "that weird course I took in between Algebra I and Algebra II". More importantly, it highlights the general notion of *how mathematics (often) advances.*

To make that last point, consider this overview of the development of exponents.

[10] In fact, it's possible to define the trig values associated with an angle with an *imaginary* component, or an "angle" that's a *matrix*, or some other fanciful thing. Doing so, however, is beyond the scope of this document.

- Introduce (strictly-positive, integer) exponents as short-hand for "repeated multiplication".
- Use intuition (in the form of intuitive counting arguments) to establish relational formulas, including
$$x^a \cdot x^b = x^{a+b} \qquad\qquad x^a/x^b = x^{a-b} \text{ for } a > b$$
- Leverage such established formulas to give meaning to expressions such as "x^0", "x^{-6}", and "$x^{\frac{1}{2}}$", which *have no inherent meaning* in the original context.
- Use intuition (in the form of intuitive "logical arithmetic") to establish relational formulas, including
$$x^b = e^{b \cdot lnx}$$
- Leverage such established formulas to give meaning to such expressions as "3^i", which *have no inherent meaning* even in the expanded context.

Our understanding of objects in a reasonably-uncomplicated realm forms the basis of relational facts, and then those relational facts form the basis of our understanding of object in a slightly-more-complictated realm; the tail, as they say, wags the dog. And this pattern is repeated throughout mathematics; witness the advent of complex numbers, fractional (even negative) dimensions and factorials, Lebesgue integration, and ... the trigonometric functions.

One good reason to stick with Quadrant I is that almost (if not absolutely) every worthwhile identity has an elegant "picture-proof" to justify it in that context. The bulk of this paper (hopefully) shows that the Fundamental Trigonograph serves as the picture for many such proofs. Others abound. For example, this article

http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

gives some picture-proofs (though not my favorite one) of the Angle-Addition Formulas[11]

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad\qquad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

I use these formulas as the springboard for expanding our angular horizons.

If we were at all squeamish about our "degenerate" configurations in the Fundamental Trigonograph, for instance, we might reasonably view our presumed values of $\sin 90°$ and $\cos 90°$ with some suspicion. However, by assigning $A = 30°$ and $B = 60°$, the above formulas *confirm* those suspect values using *irrefutable* trig values for non-degenerate angles A and B; that is, while the left-hand sides of the formulas might be "iffy", the right-hand sides are perfectly well-defined for the given A and B, and the resulting computation should remove any lingering doubt about the true nature of $\sin 90°$ and $\cos 90°$. (Likewise, the Angle-Subtraction Formulas confirm the values of $\sin 0°$ and $\cos 0°$.)

More dramatically, in a context where $\sin\theta$ and $\cos\theta$ are lengths of segments in a right triangle having an angle θ, the expressions "$\sin 120°$" and "$\cos 120°$" are laughably non-sensical. But the Angle-Addition Formula tells us that, *if "$\cos 120°$" is to equal ANYTHING, then it must be equal to* $-\frac{1}{2}$ (the result of entering $A = B = 60°$ into the right-hand side of the formula); likewise the

[11]Drifting a bit off topic, here's a "cheer" to help remember the Angle-Addition and -Subtraction Formulas:

Sine! Cosine! Sign! Cosine! Sine!
Cosine, cosine, co-sign, sine, sine!

Sprinkling in As and Bs as necessary, the lines encode the right-hand sides of the formulas for $\sin(A \pm B)$ and $\cos(A \pm B)$, where we take "sign" to indicate the *same sign* (plus or minus) as in the combined argument from the left-hand side, and "co-sign" indicates the *opposite sign* (minus or plus):

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \text{sign} : \pm \to \pm$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \text{co} - \text{sign} : \pm \to \mp$$

expression "$\sin 120°$" can be given meaning. More generally, however, having come to grips with the trig values at 90°, we can extend our understanding throughout all of Quadrant II via these computations:

$$\sin(A + 90°) = \sin A \cos 90° + \cos A \sin 90° = \cos A$$
$$\cos(A + 90°) = \cos A \cos 90° - \sin A \sin 90° = -\sin A$$

whose right-hand sides are, once more, are perfectly well-defined for acute (even right) A. And with an understanding of Quadrants I and II, we can spill into Quadrants III and IV (and beyond), arriving in the end with the traditional sign assignments and the any-angle structure of the Fundamental Trigonograph.

Of course, this Quadrant-by-Quadrant approach is inefficient, and it certainly isn't worth spending an *inordinate* amount of time on. (Some of the concepts can make for thought-provoking homework exercises.) After establishing the gist of the process, liberal hand-waving is more than appropriate. The point is to take the opportunity (if only in passing) to present some sense of the journey[12] in the development of Trig, showing it —and Mathematics in general— not as a collection of pre-fabricated definitions but an ever-lengthening chain of "what if" questions and their answers.

[12]While certainly outside the scope of a course that introduces Trigonometry to students, it's worth pointing out here that the journey continues: the Power Series Properties not only form the bridge to the realm in which "$\sin i$" and "$\cos i$" have meaning, they also pave the way for Euler's Formula and the polar representation for complex numbers and all the mathematics that comes from that.