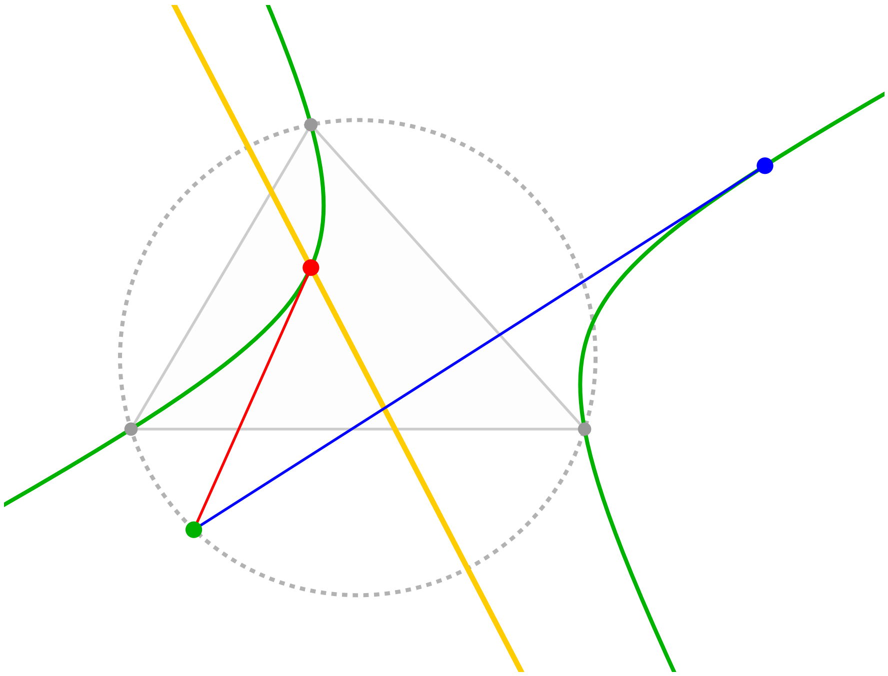

A TREATISE ON A TRIAD OF TETRADIC TRIANGLE CENTERS

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Date: August 21, 2022.

We consider three *triangle centers*[1] of a generic, non-degenerate $\triangle ABC$: Orthocenter $E := X_4$ is familiar as the point where the triangle's altitudes concur. Point $D := X_{74}$, on the circumcircle, is where the parallels at A, B, C to the *Euler line*,[2] upon reflection in corresponding angle bisectors of the triangle, concur. Finally, $F := X_{1138}$ is the unique point for which the Euler lines of $\triangle ABC$, $\triangle FBC$, $\triangle AFC$, $\triangle ABF$ are parallel.[3]

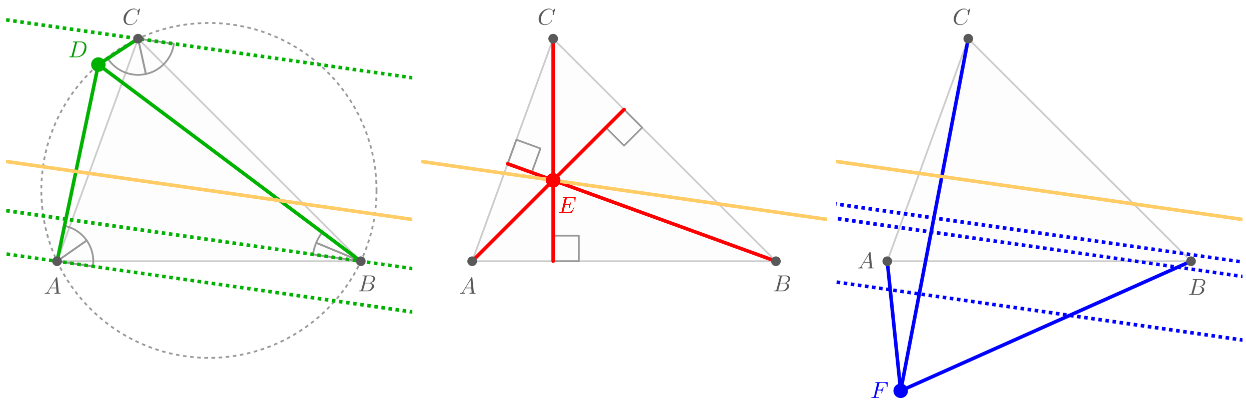

FIGURE 1. Triangle Centers $D = X_{74}$, $E = X_4$, $F = X_{1138}$

The points' disparate definitions belie a highly unusual commonality:

> *If P is a specified triangle center of $\triangle ABC$, then vertices A, B, C themselves are the corresponding centers for respective triangles $\triangle PBC$, $\triangle APC$, $\triangle ABP$.*[4]

This aspect of orthocenter E is well-known and age-old, with references dating to Archimedes,[5] and it is common and significant enough in geometric discourse to have elicited a collective term for the points A, B, C, E: *orthocentric system* (or *quadrangle* or *quadrilateral* or *group* or *set*). This note proposes generalizing that term to *centric tetrad* for points A, B, C, P exhibiting the described property, dubbing P a *tetradic center* of $\triangle ABC$.

In 2003, Floor van Lamoen [14] observed the tetradic nature of D; in 2021, this author [2] did the same for F, although "Lky" also asserted this a year earlier [15].[6]

Whether additional tetradic centers exist is an open question. Apart from a limited investigation at the end, this note doesn't address that question directly; rather, it surveys properties

[1]Clark Kimberling [12] codified the notion of a *triangle center* as a point with a fully-symmetric definition relative to its host triangle. Specifically here, a center's barycentric coordinates have the form $f(a, b, c) : f(b, c, a) : f(c, a, b)$, for some function f such that $f(a, b, c) = f(a, c, b)$ and such that f is homogenous in the triangle's side-lengths a, b, c. Kimberling's *Encyclopedia of Triangle Centers (ETC)* [13] currently documents and cross-references over *fifty thousand* instances, with the n-th one designated X_n.

[2]The Euler line contains numerous points commonly associated with a triangle; of primary interest to the current discussion: the orthocenter X_4, circumcenter X_3, and Euler infinity point X_{30} (the line's point-at-infinity). Amusingly, although the line has become a staple of triangle geometry (and features quite prominently in this note), Leonhard Euler's own interest in it seems to have begun and ended with observations about the distances between the orthocenter, circumcenter, and centroid X_2, only *implicitly* asserting the collinearity of those points. (See [16].)

[3]This is effectively the definition given by Bernard Gibert in [10]. See also Francisco Javier García Capitán [3], who observed that the locus of P that makes the Euler line of $\triangle PBC$ parallel to that of $\triangle ABC$ is a particular circumellipse, so that F is the fourth point of concurrence of the three associated circumellipses.

[4]See Section 7 for a *functional* formulation of this property.

[5]In Proposition 5 of his *Book of Lemmas*, Archimedes appears to exploit this property rather matter-of-factly to prove a result about tangent circles in an arbelos. (See Heath [11].) Nathan Altshiller-Court [1] credits Lazare Carnot [4] with re-discovering the property and calling attention to it for its own sake.

[6]"Lky" also erroneously claimed that X_{16} is tetradic.

of D, E, F and relationships among them as a triad, which may-or-may-not provide clues in the search for any tetradic siblings. Before bogging-down in formulas and equations, we'll describe two more commonalities exhibited by our triad members individually, and a key property they exhibit collectively.[7]

Euler and Brocard concurrences. We *defined* F as the point for which the Euler lines of associated triangles $\triangle FBC$, $\triangle AFC$, $\triangle ABF$, and $\triangle ABC$ itself, are parallel; that is, these four lines are *concurrent* at the Euler line's point-at-infinity, X_{30}. Also, since triad member D lies on the circumcircle of $\triangle ABC$, the Euler lines of *its* associated triangles concur at the circumcenter. Finally, it's not too difficult to show that the Euler lines of E's associated triangles concur at X_5, the *nine-point center* of $\triangle ABC$. So, such Euler concurrency is a feature common to all triad members; it is, however, not exclusive to them: the locus of all points P with this property is known to be the union of the circumcircle and the *Neuberg circumcubic*, about which we'll have more to say in Sections 3 and 5.

Frank and F. V. Morley [8] showed that the Euler lines of a point's associated triangles concur if and only if the Brocard axes of those triangles do.[8] The points of Brocard concurrency for D, E, F are, respectively, X_3 (the circumcenter), X_{52} (the orthocenter of the orthic triangle), X_{15786} (helpfully identified by *ETC* as the *intersection of the Brocard axes of the triangulation of X(1138)*).

Shadowplay. Recall that the *pedal triangle* (respectively, *cevian triangle*) of a non-vertex point P has its vertices where the perpendiculars (respectively, cevians) through P meet the side-lines of $\triangle ABC$. (Figure 2.)

Jean-Pierre Ehrmann [6] proved that E and F are the only instances of P whose pedal and cevian triangles are similar, E's *directly* (and trivially so!) and F's *indirectly.* Triad member D gets no consideration here: as with every point on the circumcircle, D's pedal triangle degenerates to the point's *Simson line* (which, for D, happens to be perpendicular to the Euler line); since its cevian triangle is non-degenerate, similarity simply isn't in play.

And yet ...

We can say that D's flat pedal triangle is similar to the *shadow* of D's cevian triangle cast in the direction of that triangle's own Euler line.[9] (Figure 3.) This, along with a slight re-thinking of Ehrmann's result, makes possible a triad-inclusive formulation:

> *Triad member P's cevian triangle, scaled in the direction of its Euler line by an appropriate factor k, is similar to P's pedal triangle. Cases* $P = D, E, F$ *correspond to* $k = 0, 1, -1$. *Moreover, each of D, E, F is the* only *non-vertex point with this property for its respective value of k.*

[7] Here and throughout, proofs are omitted. The reader may verify results via, say, intensive manipulation of barycentric coordinates (introduced in the following section).

[8] Antreas Hatzipolakis, *et al.*, [5] proved more generally for lines whose points L satisfy

$$\mathcal{L}_n : \sum_{\text{cyc}} |LA|^2 \left(|CA|^n - |AB|^n\right) = 0$$

that the $\mathcal{L}_n$ lines of a point's associated triangles concur if and only if the $\mathcal{L}_{-n}$ lines do. (The Euler line and Brocard axis correspond to $n = 2, -2$.) Vu Thanh Tung has conjectured [17] that, for each n, there is a unique point P for which the $\mathcal{L}_n$ lines of P's associated triangles are *parallel*, and that this P is tetradic. The result is true for $n = 2$, for which $P = F$. However, for $n \neq 2$, there is an entire locus of points P with the parallelism property; whether such a locus contains a unique *triangle center* —tetradic or otherwise— is not at all clear.

[9] In particular, the shadows have (signed) side-lengths in the proportion $\frac{a^2}{u_D} : \frac{b^2}{v_D} : \frac{c^2}{w_D}$, where $u_D : v_D : w_D$ are D's barycentric coordinates. (See Section 1.) The flatness of the shadows is reconfirmed by the fact that $\frac{a^2}{u_D} + \frac{b^2}{v_D} + \frac{c^2}{w_D} = 0$.

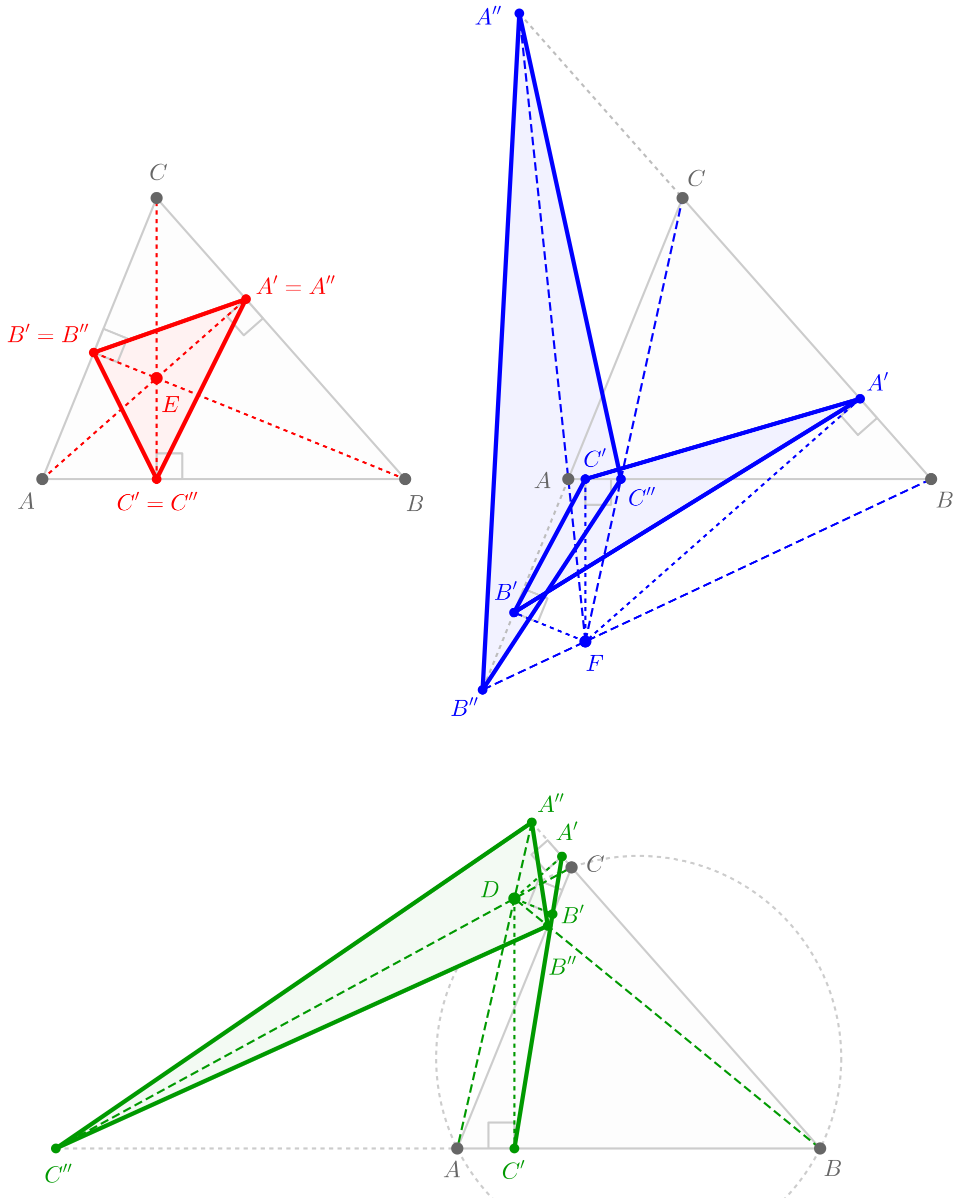

FIGURE 2. Pedal Triangles $\triangle A'B'C'$ and Cevian Triangles $\triangle A''B''C''$ of E, F, and D

The simpler formulation *"P's cevian and pedal triangles have similar Euler shadows"* is also triad-inclusive: for $P = E$ or F, the similar triangles necessarily have similar Euler shadows; for $P = D$, the pedal triangle serves as its own Euler shadow.[10] However, this property holds for a continuum of points P on a circumscribed degree-13 curve defined by this relation:

$$\sum_{\text{cyc}} \frac{(\triangle_B + \triangle_C)\,(b^2\bar{b}^2 - c^2\bar{c}^2)\,(-a^2\bar{a}^2 + b^2\bar{b}^2 + c^2\bar{c}^2)}{a^2\,\triangle_B^2\,\triangle_C^2\;-\;\bar{a}^2\,\triangle_A^2\,|\triangle ABC|^2} \;=\; 0 \qquad \bar{a} := |PA|,\ \triangle_A := |\triangle PBC|,\ \text{etc}$$

[10]The degenerate pedal triangle's circumcenter lies on the line at infinity, in the direction perpendicular to the line containing its vertices, and its centroid lies on the line of vertices, so its Euler line is perpendicular to that line.

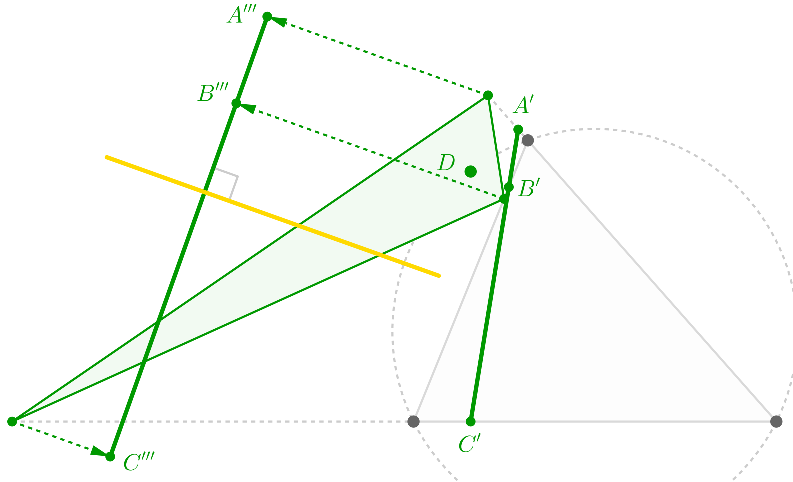

FIGURE 3. Euler Shadow $\triangle A'''B'''C'''$ of the Cevian Triangle of D

Are D, E, F the only *triangle centers* on this curve? If not, then might other centers on the curve be *tetradic*? These questions remain open.

Crosspointillism and Triadic Duality. *ETC* notes that D is the *crosspoint* of E and F, and thus that E and F are D*-cross-conjugates* of each other.[11] We also have that D is *its own* D-cross-conjugate, a fact that may seem too trivial to mention, but isn't: it's our first encounter with E and F exhibiting a kind of *duality* for which D exhibits the corresponding *self-duality*. Indeed, the notion appears often enough in this note to warrant notation for a triad member's "dual point"; we'll use this:

$$D_\star := D \qquad E_\star := F \qquad F_\star := E$$

Thus, we can say that D *is the crosspoint of* P *and* $P_\star$ *for any triad member* P; and —writing k_P for the "appropriate [scale] factor" described in *Shadowplay* above— that $k_P = -k_{P_\star}$. Even so, the reader can expect future invocations of the notion to be a bit more profound.

Randy Hutson characterizes the crosspoint P of points Q and R as the intersection of the tangent lines at Q and R to the circumconic $ABCQR$.[12] This is a keen insight in general, but especially so in the case of the triad, since $ABCEF$ is not just *any* conic: it is specifically *the unique rectangular circumhyperbola with asymptotes parallel and perpendicular to the Euler line*. *ETC* identifies this as the *Yiu hyperbola*,[13] and we'll see it throughout this note.

Hutson's insight, then, inspires the triad family portrait that serves as this note's cover image. It also suggests this not-exactly-practical triad construction:

> *Given* $\triangle ABC$*, its circumcircle, its Euler line, and its Yiu hyperbola, the line meets the hyperbola at* E*; the hyperbola's tangent at* E *meets the circle at* D*; and the "other" tangent* from D *meets the hyperbola at* F*.*

[11] *ETC*: $X(74)$ reference and Glossary "crosspoint" entry. Algebraically, the barycentric coordinates of D, E, F satisfy the relation in Footnote 16. Geometrically, D is the point of concurrence of the lines through $AE \cap BF$ and $AF \cap BE$, $BE \cap CF$ and $BF \cap CE$, and $CE \cap AF$ and $CF \cap AE$.

[12] As credited in *ETC* Glossary entry for "crosspoint"; 10 September, 2012. Note that we can say more: The tangent at P to conic $ABCPQ$ (respectively, $ABCPR$) meets the non-R point where line QR meets conic $ABCPR$ (respectively, the non-Q point where QR meets $ABCPQ$).

[13] See *ETC*'s X(5627) entry, referencing a puzzle by Paul Yiu [18].

1. Barycentric Basics

Barycentric $u : v : w$ coordinates of the triad points are as follows:[14]

$$D := \frac{a^2}{(a^2-b^2+c^2)(a^2+b^2-c^2)-a^2(-a^2+b^2+c^2)} : \cdots : \cdots$$

$$(1) \qquad E := \frac{1}{-a^2+b^2+c^2} : \frac{1}{-b^2+c^2+a^2} : \frac{1}{-c^2+a^2+b^2}$$

$$F := \frac{a^2b^2c^2}{(-a^2+b^2+c^2-bc)^2(-a^2+b^2+c^2+bc)^2+9b^2c^2(c^2-a^2)(a^2-b^2)} : \cdots : \cdots$$

Of course, as *homogeneous* coordinates, these are determined only up to a non-zero multiplied constant. (We could, therefore, divide F's coordinates through by $a^2b^2c^2$, but we'll leave them as-is so that all coordinates of all triad points have degree -2 as rational functions of a, b, c.) For the sake of specificity, any future reference to a particular coordinate *means* one of the expressions indicated in (1); this gives us the freedom to write non-homogeneous equations such as these:

$$\frac{u_D+v_D+w_D}{u_Dv_Dw_D} = -\frac{\rho^2\tau^2}{\mu^2} \qquad \frac{u_E+v_E+w_E}{u_Ev_Ew_E} = \tau^2 \qquad \frac{u_F+v_F+w_F}{u_Fv_Fw_F} = -\frac{3\rho^4\tau^2}{\mu^4}$$

where

$$\tau := \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} = 4\,|\triangle ABC|$$

$$\rho := \sqrt{-\left(a^2\,\delta_b\delta_c + b^2\,\delta_c\delta_a + c^2\,\delta_a\delta_b\right)} \qquad \mu := abc$$

$$\delta_a := b^2-c^2 \qquad \delta_b := c^2-a^2 \qquad \delta_c := a^2-b^2 \qquad \delta := \delta_a\delta_b\delta_c$$

The coordinate expressions satisfy a litany of identities, such as these cyclic sums:[15]

$$(2) \qquad \sum_{\text{cyc}} \frac{a^2}{u_D} = 0 \qquad \sum_{\text{cyc}} \frac{a^2}{u_E} = \tau^2 \qquad \sum_{\text{cyc}} \frac{a^2}{u_F} = -\frac{\rho^2\tau^2}{\mu^2}$$

$$(3) \qquad \sum_{\text{cyc}} \frac{\delta_a}{u_D} = -\frac{\tau^2\delta}{\mu^2} \qquad \sum_{\text{cyc}} \frac{\delta_a}{u_E} = 0 \qquad \sum_{\text{cyc}} \frac{\delta_a}{u_F} = \frac{6\tau^2\delta}{\mu^2}$$

$$(4) \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_D} = -6\delta \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_E} = 2\delta \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_F} = 18\delta$$

$$(5) \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_D^2} = -\frac{\tau^4\delta}{\mu^2} \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_E^2} = \sum_{\text{cyc}} \frac{a^2\delta_a}{u_F^2} = 0$$

$$(6) \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_Eu_F} = \frac{2\tau^4\delta}{\mu^2} \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_Du_E} = \sum_{\text{cyc}} \frac{a^2\delta_a}{u_Du_F} = 0$$

$$(7) \qquad \sum_{\text{cyc}} \frac{1}{u_Du_E}\left(\frac{1}{v_E^2}-\frac{1}{w_E^2}\right) = \sum_{\text{cyc}} \frac{1}{u_Du_F}\left(\frac{1}{v_F^2}-\frac{1}{w_F^2}\right) = 0$$

[14] Omitted v and w coordinates for any triangle center derive from u via cyclic substitutions $a \to b \to c \to a$ and $u_P \to v_P \to w_P \to u_P$. The same substitutions govern cyclic sums.

[15] Intriguingly, the E-, D-, and F-sums in (4) are in geometric progression $1 : -3 : 9$, as are the appropriately-ordered "left-sum over right-sum" ratios in (9). The latter calculations underlie the identical progression in (16).

(8)
$$\sum_{\text{cyc}} \frac{a^2}{u_D u_E}\left(\frac{b^2}{v_E^2} - \frac{c^2}{w_E^2}\right) \;=\; \sum_{\text{cyc}} \frac{a^2}{u_D u_F}\left(\frac{b^2}{v_F^2} - \frac{c^2}{w_F^2}\right) \;=\; 0$$

(9)
$$\sum_{\text{cyc}} \frac{a^2}{u_D^2 u_E}\left(\frac{1}{v_D w_E} - \frac{1}{v_E w_D}\right) = -\frac{\tau^6\delta}{\mu^2} \qquad \sum_{\text{cyc}} \frac{\delta_a}{u_D u_E}\left(\frac{1}{v_D w_E} - \frac{1}{v_E w_D}\right) = -\frac{\tau^2}{u_D v_D w_D}$$

$$\sum_{\text{cyc}} \frac{a^2}{u_D^2 u_F}\left(\frac{1}{v_D w_F} - \frac{1}{v_F w_D}\right) = \frac{9\tau^6\rho^4\delta}{\mu^6} \qquad \sum_{\text{cyc}} \frac{\delta_a}{u_D u_F}\left(\frac{1}{v_D w_F} - \frac{1}{v_F w_D}\right) = \frac{\tau^2\rho^4}{\mu^4\, u_D v_D w_D}$$

$$\sum_{\text{cyc}} \frac{a^2}{u_D u_E u_F}\left(\frac{1}{v_E w_F} - \frac{1}{v_F w_E}\right) = -\frac{6\tau^6\rho^2\delta}{\mu^4} \qquad \sum_{\text{cyc}} \frac{\delta_a}{u_E u_F}\left(\frac{1}{v_E w_F} - \frac{1}{v_F w_E}\right) = \frac{2\tau^2\rho^2}{\mu^2\, u_D v_D w_D}$$

(10)
$$\sum_{\text{cyc}} \frac{a^2}{u_D u_E^2}\left(\frac{1}{v_D w_E} - \frac{1}{v_E w_D}\right) = \frac{\tau^6\delta}{\mu^2} \qquad \sum_{\text{cyc}} \frac{a^2}{u_D u_E u_F}\left(\frac{1}{v_D w_E} - \frac{1}{v_E w_D}\right) = -\frac{3\tau^6\rho^2\delta}{\mu^4}$$

$$\sum_{\text{cyc}} \frac{a^2}{u_D u_E u_F}\left(\frac{1}{v_D w_F} - \frac{1}{v_F w_D}\right) = -\frac{\tau^6\delta}{\mu^6}\,\xi \qquad \sum_{\text{cyc}} \frac{a^2}{u_D u_F^2}\left(\frac{1}{v_D w_F} - \frac{1}{v_F w_D}\right) = \frac{3\tau^6\rho^2\delta}{\mu^8}\,\xi$$

$$\sum_{\text{cyc}} \frac{a^2}{u_E^2 u_F}\left(\frac{1}{v_E w_F} - \frac{1}{v_F w_E}\right) = \frac{6\tau^6\rho^2\delta}{\mu^4} \qquad \sum_{\text{cyc}} \frac{a^2}{u_E u_F^2}\left(\frac{1}{v_E w_F} - \frac{1}{v_F w_E}\right) = -\frac{2\tau^6\delta}{\mu^6}\,\xi$$

$$\xi := \left(\mu + \frac{a}{u_E} + \frac{b}{v_E} + \frac{c}{w_E}\right)\left(\mu + \frac{a}{u_E} - \frac{b}{v_E} - \frac{c}{w_E}\right)\left(\mu - \frac{a}{u_E} + \frac{b}{v_E} - \frac{c}{w_E}\right)\left(\mu - \frac{a}{u_E} - \frac{b}{v_E} + \frac{c}{w_E}\right)$$

Additional identities include these symmetric relations:[16]

(11)
$$\frac{1}{u_D}\left(\frac{1}{v_E w_F} + \frac{1}{v_F w_E}\right) = \frac{1}{v_D}\left(\frac{1}{w_E u_F} + \frac{1}{w_F u_E}\right) = \frac{1}{w_D}\left(\frac{1}{u_E v_F} + \frac{1}{u_F v_E}\right) = \frac{2}{u_D v_D w_D}$$

(12)
$$\frac{u_D}{a^2\delta_a}\left(\frac{1}{v_E w_F} - \frac{1}{v_F w_E}\right) = \frac{v_D}{b^2\delta_b}\left(\frac{1}{w_E u_F} - \frac{1}{w_F u_E}\right) = \frac{w_D}{c^2\delta_c}\left(\frac{1}{u_E v_F} - \frac{1}{u_F v_E}\right) = -\frac{2\tau^2}{\mu^2}$$

(13)
$$\frac{u_E}{a^2\delta_a}\left(\frac{1}{v_D w_E} - \frac{1}{v_E w_D}\right) = \frac{v_E}{b^2\delta_b}\left(\frac{1}{w_D u_E} - \frac{1}{w_E u_D}\right) = \frac{w_E}{c^2\delta_c}\left(\frac{1}{u_D v_E} - \frac{1}{u_E v_D}\right) = \frac{\tau^2}{\mu^2}$$

(14)
$$\frac{u_F}{a^2\delta_a}\left(\frac{1}{v_D w_F} - \frac{1}{v_F w_D}\right) = \frac{v_F}{b^2\delta_b}\left(\frac{1}{w_D u_F} - \frac{1}{w_F u_D}\right) = \frac{w_F}{c^2\delta_c}\left(\frac{1}{u_D v_F} - \frac{1}{u_F v_D}\right) = -\frac{\tau^2}{\mu^2}$$

(15)
$$a^2\delta_a\left(\frac{1}{u_E u_F} - \frac{1}{u_D^2}\right) = b^2\delta_b\left(\frac{1}{v_E v_F} - \frac{1}{v_D^2}\right) = c^2\delta_c\left(\frac{1}{w_E w_F} - \frac{1}{w_D^2}\right) = \frac{\tau^4\delta}{\mu^2}$$

Many results in this note trace back to these relations.

[16] Relation (11) in proportional form

$$u_D : v_D : w_D \quad \propto \quad \frac{1}{v_E w_F} + \frac{1}{v_F w_E} \;:\; \frac{1}{w_E u_F} + \frac{1}{w_F u_E} \;:\; \frac{1}{u_E v_F} + \frac{1}{u_F v_E}$$

is precisely the barycentric definition of D as the crosspoint of E and F.

2. Geometric Generalities in $\triangle DEF$

The area of $\triangle DEF$ is given by

$$\frac{|\triangle DEF|}{|\triangle ABC|} = \begin{vmatrix} u_D & v_D & w_D \\ u_E & v_E & w_E \\ u_F & v_F & w_F \end{vmatrix} = \frac{4\tau^6}{a^4b^4c^4}\,|\delta_a\delta_b\delta_c\; u_Ev_Ew_E\; u_Fv_Fw_F|$$

The barycentric equations of the side-lines are[17]

$$\overleftrightarrow{EF} : \sum_{cyc} \frac{a^2\delta_a}{u_Du_Eu_F}\,u = 0 \qquad \overleftrightarrow{DE} : \sum_{cyc} \frac{a^2\delta_a}{u_Du_E^2}\,u = 0 \qquad \overleftrightarrow{DF} : \sum_{cyc} \frac{a^2\delta_a}{u_Du_F^2}\,u = 0$$

We can exploit the triad's notion of duality by expressing all three equations in the common form

$$\overleftrightarrow{PQ} : \sum_{cyc} \frac{a^2\delta_a}{u_Pu_Qu_{R_\star}}u = 0 \qquad \{P,Q,R\} = \{D,E,F\}$$

The lengths of the sides of $\triangle DEF$ are given by

$$|EF|^2 = \frac{2a^4b^4c^4}{9\tau^2\rho^8}\sum_{\text{cyc}}\frac{a^4\delta_a^2}{u_D^2u_Eu_F^2} \qquad 9|EF|^2 - 4|DE|^2 \;=\; 8\frac{\tau^4\delta_a^2\delta_b^2\delta_c^2}{\rho^6} \;=\; 4|FD|^2 - |EF|^2$$

From this, we can deduce that $5|EF|^2 = 2|FD|^2 + 2|DE|^2$ and thus that, by Apollonius' Theorem, the D-median has length $|EF|$. Some follow-on consequences appear below, where M is the midpoint of DE, N is the midpoint of DM (and lies on the Euler line), and O_D and O_F (which we'll encounter again in Section 4) are the respective midpoints of MN and DE; also, references to the *Euler line* and *circumcircle* indicate those elements of host triangle $\triangle ABC$. (Figure 4.)

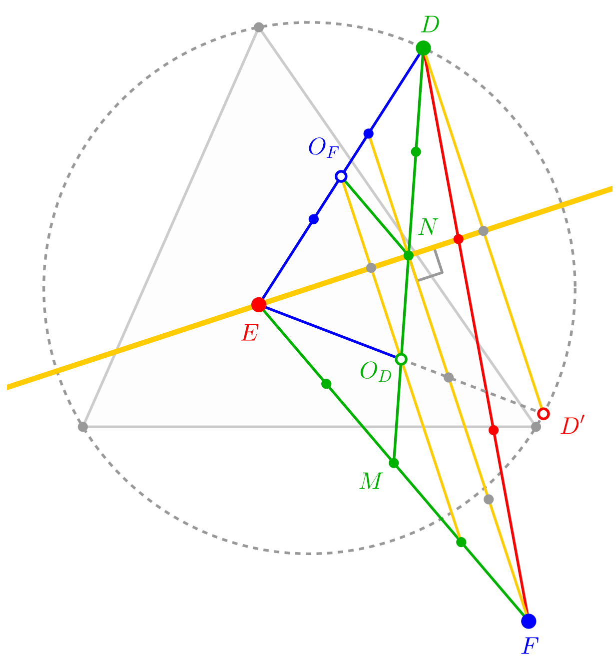

Figure 4. Geometry of $\triangle DEF$, showing the circumcircle and Euler line of $\triangle ABC$

[17]Interestingly, D *disappears* from the area formula, but *appears* in the equation for the line through the *other two* triad members.

- The E- and F-cevians through N (the former being the Euler line itself) are perpendicular, and they trisect edges DF and EF.
- N quadrisects those cevians, and
$$|EN| = \frac{a^2b^2c^2}{3\rho^3\tau\ |u_Dv_Dw_D|} \qquad |FN| = \frac{\tau^2}{\rho^3}\ |\delta_a\delta_b\delta_c|$$
- Writing θ_{PQ} for the (signed) acute angle made by side-line PQ and the Euler line, we have[18]
$$\tan\theta_{DE} : \tan\theta_{EF} : \tan\theta_{FD} \ \propto\ 1 : -3 : 9 \tag{16}$$
In particular, for an appropriate orientation,
$$\tan\theta_{DE} = \tau^3\,\frac{\delta_a\delta_b\delta_c\ u_Dv_Dw_D}{a^2b^2c^2}$$
- O_D and O_F are mutual reflections in the Euler line, and
$$|O_DO_F| = \tfrac{1}{2}|FN|$$
This is also the distance from D to the Euler line.
- The reflection of E in O_D (point D' in the figure) is the reflection of D in the Euler line; hence it lies on the circumcircle. (It is in fact the fourth intersection of the circumcircle with the circumhyperbola $ABCEF$; the point is Kimberling's X_{477}.)

We can glean from these facts a concrete construction of the triad: First, construct $D' := X_{477}$.[19] Then D is the reflection of D' in the Euler line. Point F's distance from the Euler line is $|DD'|$, and its distance from line DD' is one-third E's distance to that line. (Figure 5.)

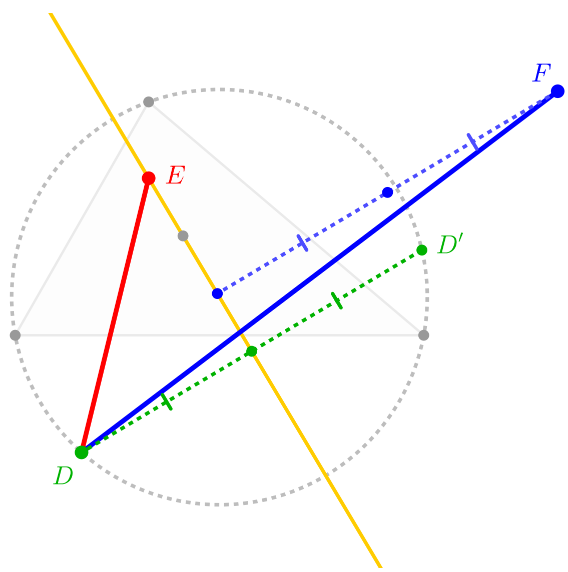

FIGURE 5. A construction of the triad

[18]The proportion can be gleaned from Figures 4 and 5. Individual values arise from the sums (9) and this formula:
$$\tau^{3/2}\ \tan\theta_{PQ} = \left(\sum_{\text{cyc}} \frac{a^2}{u_Du_Pu_Q}\left(\frac{1}{v_Pw_Q} - \frac{1}{v_Qw_P}\right)\right) \Big/ \left(\sum_{\text{cyc}} \frac{\delta_a}{u_Pu_Q}\left(\frac{1}{v_Pw_Q} - \frac{1}{v_Qw_P}\right)\right)$$

[19]For instance, let the Euler line meet side-lines AB and AC at B' and C', then the line through A and the circumcenter of $\triangle AB'C'$ meets the circumcircle at X_{110}. (This is Randy Hutson's 6th construction of X_{110}. See *ETC*'s X(110) entry.) The line through this point, parallel to the Euler line, meets the circumcircle again at D'.

3. Conjugate Considerations

Let P^+ and P^- denote the *isogonal* and *isotomic conjugates*[20] of a point P. The conjugates of D, E, F, and half of the conjugates of *those* conjugates, are documented triangle centers:

$D^+ := X_{30}$	(Euler infinity point)	$D^- := X_{3260}$	$(D^+)^- := X_{1494}$	$(D^-)^+ :=$ –
$E^+ := X_3$	(circumcenter)	$E^- := X_{69}$	$(E^+)^- := X_{264}$	$(E^-)^+ := X_{25}$
$F^+ := X_{399}$	(Parry reflection point)	$F^- := X_{1272}$	$(F^+)^- :=$ –	$(F^-)^+ :=$ –

Correspondingly, we use $\pm$ superscripts on individual barycentric coordinates of such conjugates:[21]

$$P^+ = u^+ : v^+ : w^+ \;:=\; \frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w} \qquad\qquad P^- = u^- : v^- : w^- \;:=\; \frac{1}{u} : \frac{1}{v} : \frac{1}{w}$$

As with the coordinates of D, E, F themselves, we ignore homogeneity in defining $u_P^\pm$, $v_P^\pm$, $w_P^\pm$ in terms of the specific expressions u_P, v_P, w_P, allowing us to write non-homogeneous equations such as these "flattened" forms of (2) and (3):

$$u_D^+ + v_D^+ + w_D^+ \;=\; 0 \qquad u_E^+ + v_E^+ + w_E^+ \;=\; \tau \qquad u_F^+ + v_F^+ + w_F^+ \;=\; -\frac{\rho^2\tau^2}{\mu^2}$$

$$\delta_a u_D^- + \delta_b v_D^- + \delta_c w_D^- \;=\; -\frac{\tau^2\delta}{\mu^2} \qquad \delta_a u_E^- + \delta_b v_E^- + \delta_c w_E^- \;=\; 0 \qquad \delta_a u_F^- + \delta_b v_F^- + \delta_c w_F^- \;=\; \frac{6\tau^2\delta}{\mu^2}$$

Concurrences. Triad points and their isoconjugates determine numerous concurrent lines.

- Lines DD^-, EE^-, FF^- (ie, the three lines of the form PP^- for triad point P) concur at D^-. (Figure 6.) This property follows from (7), and it places our triad (and their isotomic conjugates) on the circumcubic K_- described in Section 5.

  Line EE^- also contains $(E^+)^-$, and FF^- also contains $(F^+)^-$.

- Lines DD^+, EE^+, FF^+ (ie, the three lines PP^+) concur at D^+, which is to say: they are parallel to the Euler line; of course, EE^+ *is* the Euler line. (Figure 6.) This follows from (8), and it places the triad (and their isogonal conjugates) on the circumcubic K_+. (Section 5.)

  Line EE^+ also contains $(E^-)^+$, and FF^+ also contains $(F^-)^+$.

- Lines D^+D^+, D^-D^-, E^+F^+, E^-F^- (ie, $P^\pm P_\star^\pm$) concur at D. (Figure 7.) Here, we interpret D^+D^+ and D^-D^- as tangent lines to the circumcubics K_+ and K_-. (Tangent line D^+D^+ is the asymptote of K_+.)

  E^+F^+ also contains $(D^-)^+$, and E^-F^- also contains $(D^+)^-$.

- Lines D^+D^-, E^+F^-, F^+E^- (ie, $P^\pm P_\star^\mp$) concur at a point not currently indexed in *ETC*; we'll call it Z. (Figure 8, left.) It has barycentric coordinates

  $$\delta_b v_D(u_E^- v_F^+ - v_E^- u_F^+) \;-\; \delta_c w_D(w_E^- u_F^+ - u_E^- w_F^+) \;:\; \cdots \;:\; \cdots$$

  Line $(D^+)^-(D^-)^+$ also contains Z.

[20] Respective cevians through P and through P^+ are mutual reflections in the corresponding angle bisectors of $\triangle ABC$. The points where respective cevians through P and those through P^- meet the side-lines of $\triangle ABC$ are mutual reflections in the midpoints of the corresponding sides.

[21] Caveat: In the *trilinear* coordinate system, say $P = h : j : k$, the *isogonal* conjugate coordinates involve simple reciprocation, $P^+ = \frac{1}{h} : \frac{1}{j} : \frac{1}{k}$, so that the isogonal conjugate of P is sometimes denoted P^{-1}. (In trilinear coordinates, $P^- = \frac{1}{a^2h} : \frac{1}{b^2j} : \frac{1}{c^2k}$.) Hopefully, this note's barycentric bias will not cause confusion.

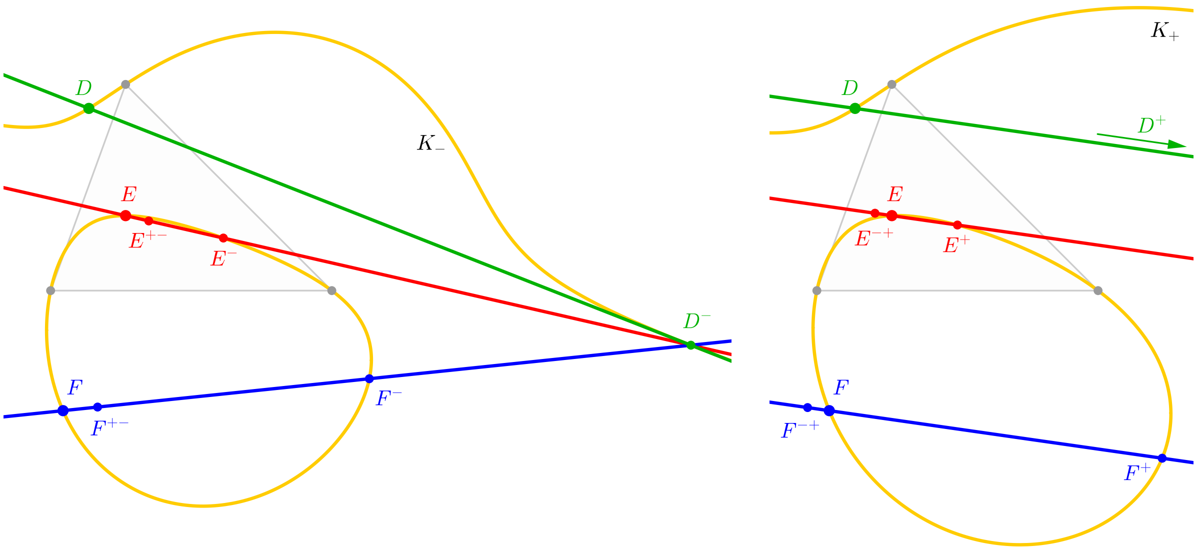

FIGURE 6

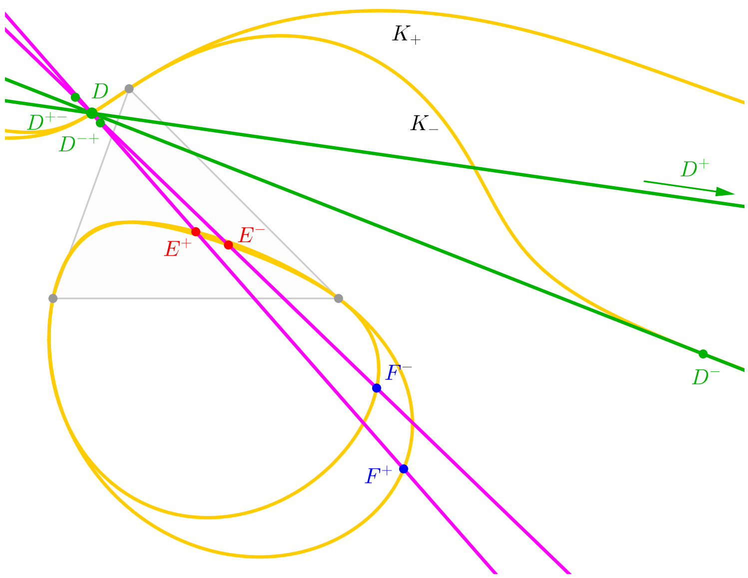

FIGURE 7

- Circles $\bigcirc EE^+E^-$ and $\bigcirc FF^+F^-$ (but not $\bigcirc DD^+D^-$) meet the triangle's circumcircle at the Steiner Point, $S := X_{99}$. (Figure 8, right.) The locus of all P —including the centroid, incenter, and excenters— such that $\bigcirc PP^+P^-$ contains S is the Stammler circumquartic (Gibert's Q_6):[22]

$$\frac{a^2\delta_a}{u^2} + \frac{b^2\delta_b}{v^2} + \frac{c^2\delta_c}{w^2} = 0$$

Line DE^-F^- also contains S.

[22]That E and F satisfy this equation, and that D does not, is stated previously in relation (5).

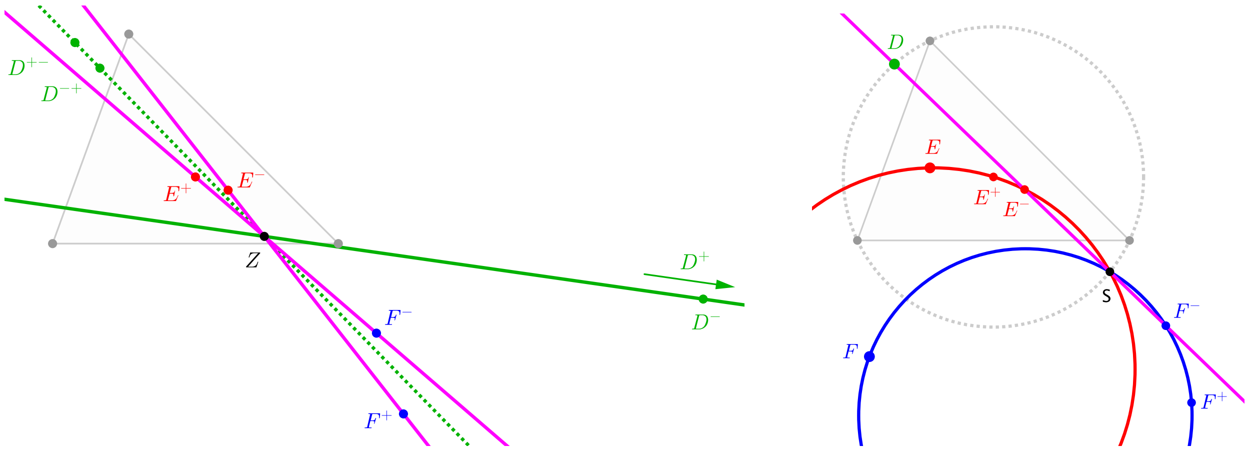

FIGURE 8

Arbitrary Isoconjugation. Kimberling[23] generalizes isogonal and isotomic conjugation via the notion of *isoconjugation* with respect to a given triangle center Q. The operation takes point P to a point we'll denote P°, defined by[24]

$$P^\circ = u_P^\circ : v_P^\circ : w_P^\circ := \frac{a^3}{u_P u_Q} : \frac{b^3}{v_P v_Q} : \frac{c^3}{w_P w_Q}$$

Thus, P^+ and P^- are isoconjugates of P with respect to $X_1 = a : b : c$ (the incenter) and $X_{31} = a^3 : b^3 : c^3$. One can show that, lines DD°, EE°, FF° concur if and only if Q lies on the line[25] X_1X_{31}, which has equation

$$\frac{u\delta_a}{a} + \frac{v\delta_b}{b} + \frac{w\delta_c}{c} = 0 \tag{17}$$

For such Q, we can say further

- Lines DD°, EE°, FF° (that is, lines PP°) concur at D°. This places the triad on the circumcubic $K_\circ$ described in Section 5.
- Lines $D^\circ D^\circ$ and $E^\circ F^\circ$ (lines $P^\circ P_\star^\circ$) meet at D. Here, $D^\circ D^\circ$ is the tangent at D° to $K_\circ$ above.
- If $P^\times$ is the isoconjugate of P relative to *some other* point R on line X_1X_{31}, then $D^\circ D^\times$, $E^\circ F^\times$, $E^\times F^\circ$ (lines $P^\circ P_\star^\times$) concur at a point with barycentric coordinates
$$\frac{a^3}{u_E^2 u_F^2 u_Q^2 u_R^2}\begin{pmatrix} u_E u_F (v_E^- w_F^- - w_E^- v_F^-)(u_Q^2 v_Q^- w_Q^- - u_R^2 v_R^- w_R^-) \\ -u_Q u_R (v_Q^- w_R^- - w_Q^- v_R^-)(u_E^2 v_E^- w_E^- - u_F^2 v_F^- w_F^-) \end{pmatrix} : \cdots : \cdots$$
Line $(D^\circ)^\times (D^\times)^\circ$ also contains this point.
- In general, $\bigcirc EE^\circ E^\times$ and $\bigcirc FF^\circ F^\times$ do not concur with $\bigcirc ABC$.
- As Q varies along X_1X_{31}, the loci of D°, E°, F° are the circumconics $ABCEF$ (the Yiu hyperbola), $ABCDE$, $ABCDF$. (In the notation of the next section, P° lies on conic $\Gamma_{P_\star}$.)

[23] See the *ETC* Glossary "isoconjugate" entry.

[24] In trilinear coordinates, with $P = h : j : k$, the definition is a bit cleaner: $P^\circ := \frac{1}{hh_Q} : \frac{1}{jj_Q} : \frac{1}{kk_Q}$.

[25] *ETC*'s "Central Lines" table identifies this line as L_{661}, the trilinear pole of X_{662}, and currently lists 149 triangle centers on it.

4. Circumconic Circumstances

We'll begin with some facts about families of circumconics of $\triangle ABC$ that pass through one of our triad members.

- Any circumconic through E is a rectangular hyperbola; conversely, any rectangular circum-hyperbola contains E.
- The center of a circum-hyperbola through E lies on the nine-point circle; specifically, it is the midpoint of E and the hyperbola's fourth intersection with the circumcircle of $\triangle ABC$.
- The circumconics through D (as with any fixed point on the circumcircle) have parallel axes of symmetry. If θ is an angle formed by the major/transverse axis and the Euler line, then
$$\cos 2\theta = \frac{\tau^3\, \delta_a \delta_b \delta_c}{2\rho^3\, abc}$$
- The center of a circumconic through D lies on the necessarily-rectangular hyperbola through the circumcenter, the midpoints of the sides, and the midpoint of DE.[26]
- For circumconics through F, neither eccentricity nor axis direction are constant, but we have
$$(1 - 2\cos 2\theta) \cdot \text{eccentricity}^2 = 2$$
where θ is an angle made by the major/transverse axis and the Euler line.

Now, let's consider three specific circumconics: Define Γ_P, for triad member P, to be the circumconic through *the other two* triad members (Figure 9); the conic's barycentric equation, using our duality notation, is:
$$\Gamma_P : \sum_{cyc} \frac{a^2 \delta_a}{u u_{P_\star}} = 0$$
Combining properties from above:

- Γ_D —earlier dubbed the *Yiu hyperbola* and featured in the triad family portrait (Figure **??**)—is a rectangular hyperbola whose transverse axis makes a 45° angle with the Euler line; thus, its asymptotes are parallel and perpendicular to that line.
- Γ_F —known in the literature as the *Jerabek hyperbola*— is a rectangular hyperbola with axes of symmetry parallel to those of Γ_E.

Bonus fact about Γ_D:

- If P is a point on Γ_D, and A', B', C' are reflections of P in the respective lines parallel to the Euler line through A, B, C, then lines AA', BB', CC' concur at another point of Γ_D; namely, the reflection of P in the conic's center.

  When $P = E$, the point of concurrence X_{477}, the fourth point where the Γ_D meets the circumcircle. When $P = F$, this point is X_{5627}, which *ETC* identifies as the Yiu Reflection Point (which inspired the name *Yiu hyperbola* for Γ_D.) In either case, the tangent to Γ_D at P is parallel to the line $DP_\star$.

[26] Since the circumcenter of $\triangle ABC$ is the orthocenter of the midpoint triangle, this conic-of-centers is a rectangular hyperbola. The conic-of-centers for circumconics through *any* fixed P passes through the midpoints of the sides; the conic-of-centers for *any* fixed P on the circumcircle passes through the circumcenter.

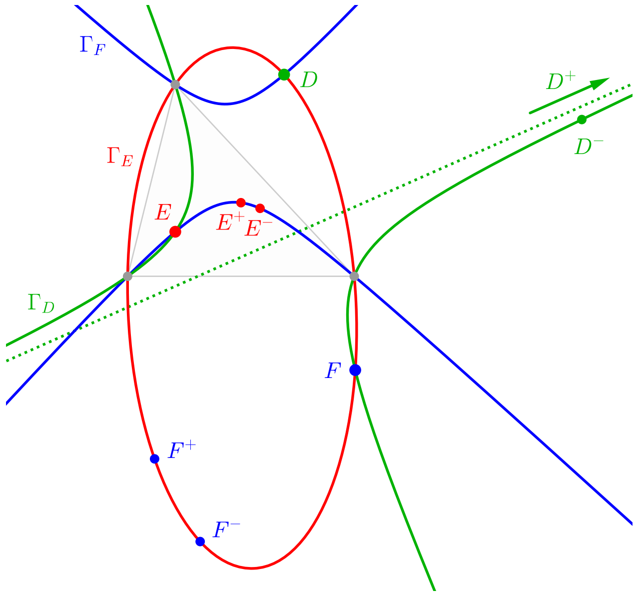

FIGURE 9. Circumconics $\Gamma_D := ABCEF$, $\Gamma_E := ABCFD$, $\Gamma_F := ABCDE$

Let O_P be the center of Γ_P. Some properties of O_D and O_F appear in Section 2.

- O_F (X_{125}) is the midpoint of DE. Its barycentric coordinates are:
$$\frac{\delta_a^2}{u_E} : \frac{\delta_b^2}{v_E} : \frac{\delta_c^2}{w_E}$$
- O_D (X_{3258}) is the reflection of O_F in the Euler line.
$$\frac{\delta_a}{u_D}\left(\frac{1}{v_F}-\frac{1}{w_F}\right) : \frac{\delta_b}{v_D}\left(\frac{1}{w_F}-\frac{1}{u_F}\right) : \frac{\delta_c}{w_D}\left(\frac{1}{u_F}-\frac{1}{v_F}\right)$$
- O_E is not (yet!) in *ETC*.
$$\frac{1}{u_F}\left(\frac{1}{v_F}-\frac{1}{w_F}\right)^2 : \frac{1}{v_F}\left(\frac{1}{w_F}-\frac{1}{u_F}\right)^2 : \frac{1}{w_F}\left(\frac{1}{u_F}-\frac{1}{v_F}\right)^2$$

Abusing notation to indicate component-wise arithmetic on the conic centers' coordinates, we have
$$\frac{O_D^2}{O_E\cdot O_F} = \frac{u_E u_F}{u_D^2} : \frac{v_E v_F}{v_D^2} : \frac{w_E w_F}{w_D^2} \qquad \left(= \frac{u_E u_{E_\star}}{u_D u_{D_\star}} : \frac{v_E v_{E_\star}}{v_D v_{D_\star}} : \frac{w_E w_{E_\star}}{w_D w_{D_\star}}\right)$$

Finally, recall that the isotomic (or isogonal) conjugate of a line is a circumconic, and conversely. Therefore, having previously observed (Section 3) that lines E^-F^-, D^-F^-, D^-E^- contain D, F, E, respectively, we know that Γ_D, Γ_E, Γ_F contain D^-, F^-, E^-; likewise, previous observations imply that they contain D^+, F^+, E^+. Succinctly,

$$\Gamma_P \textit{ contains } P_\star^- \textit{ and } P_\star^+.$$

Some consequences of this fact are explored in Section 5.

Three more conics. For P a member of the triad, define Φ_P as the circumconic through P whose tangent line at P is perpendicular to the Euler line. Equations are as follows:

$$\Phi_D : \sum_{\text{cyc}} \frac{a^2 u_D}{u_F\, u} \left(\delta_b \delta_c \tau^2 u_F - a^2 b^2 c^2\right) = 0 \qquad \Phi_P : \sum_{\text{cyc}} \frac{a^2 u_P}{u_D\, u} = 0 \quad (\text{for } P = E, F)$$

Simplicity clearly disfavors D (as does Figure 10), and we can say more about the other cases: Φ_P and line DP meet the circumcircle at a point M_P given by[27]

$$M_P := \frac{u_P^2}{\delta_a} : \frac{v_P^2}{\delta_b} : \frac{w_P^2}{\delta_c} \quad (\text{for } P = E, F) \qquad M_E = X_{107}$$

A bit more about Φ_E and Φ_F is said (and shown) in Section 5's discussion of circumcubic $K_{\varnothing}$.

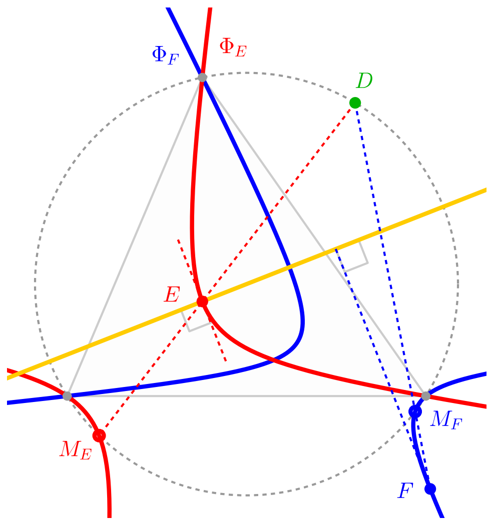

FIGURE 10. Circumconics Φ_E and Φ_F

5. CIRCUMCUBIC CURIOSITIES

Circumcubics K_+ and K_-. Bernard Gibert's catalogue of cubics [9] opens with the isogonally-self-conjugate *Neuberg cubic*,[28] K_1, herein denoted K_+. The isotomically-self-conjugate K_{279} is denoted K_-. The equations are as follows:

$$K_+ : \sum_{\text{cyc}} \frac{u}{u_D} \left(\frac{v^2}{b^2} - \frac{w^2}{c^2}\right) = 0 \qquad K_- : \sum_{\text{cyc}} \frac{u}{u_D} \left(v^2 - w^2\right) = 0 \qquad K_\pm : \sum_{\text{cyc}} \frac{u}{u_D} \left(\frac{v}{v^\pm} - \frac{w}{w^\pm}\right) = 0$$

The curves are the loci of a point P such that line PP^+ —respectively, PP^-— contains D. As noted in Section 3, and depicted in Figures 6 and 7, each contains the triad (and the appropriate conjugates thereof).

[27]While $M_E = X_{107}$, point M_F is not in *ETC*.

[28]Gibert notes that the cubic was first introduced in Joseph Jean Baptiste Neuberg's 1884 paper "Mémoire sur le tétraédre" in *Mémoires de l'Académie de Belgique*

- Barring degeneracies in the triangle, D, E, F are the *only* (non-vertex) points on K_- such that the K_- for each of $\triangle PBC$, $\triangle APC$, $\triangle ABP$ is also a circumcubic of $\triangle ABC$.[29] Thus, this property *characterizes* our triad.

- *Every* (non-vertex) point P on K_+ is such that the K_+ of each of $\triangle PBC$, $\triangle APC$, $\triangle ABP$ is also a circumcubic of $\triangle ABC$; so, this property *does not* characterize the triad.[30]

Circumcubic $K_\circ$. As of this writing, Gibert's catalogue documents only K_+ and K_- as irreducible circumcubics containing the triad. However, invoking Section 3's "arbitrary isoconjugation" yields a continuum of such cubics that also pass through associated isoconjugates D°, E°, F°. The equation, for isoconjugation with respect to Q on X_1X_{31}, is given by

$$K_\circ : \sum_{\text{cyc}} \frac{u}{u_D}\left(\frac{v^2 v_Q}{b^3} - \frac{w^2 w_Q}{c^3}\right) = 0 \tag{18}$$

Since the D° for each Q necessarily lies on the Yiu hyperbola, Γ_D, we can simplify the definition of $K_\circ$ by *taking* D° on that hyperbola and *defining* E° and F° via $u_D u_D^\circ = u_E u_E^\circ = u_F u_F^\circ$, etc. With this approach, we can write the equation in a form directly comparable to the those given above for K_+ and K_-.

$$K_\circ : \sum_{\text{cyc}} \frac{u}{u_D}\left(\frac{v^2}{v_D v_D^\circ} - \frac{w^2}{w_D w_D^\circ}\right)$$

Cubic $K_\circ$ exhibits various features of K_- and K_+. In addition to those mentioned in Section 3, and the decomposition described below, we note that the tangents at A, B, C concur at D. Also, the tangent at D and the line EF concur with the tangents at the non-vertex points of intersection with the triangle's side-lines at another point on the cubic: namely, the D°-Ceva-conjugate[31] of D.

A trio of degenerate cases, one for each vertex, are worth mentioning. For vertex A (likewise, B and C), we take D° as the *other* point where line AD meets the Yiu hyperbola. Then $K_\circ$ reduces to the union of AD and the (non-circum)conic $BCEE^\circ FF^\circ$. (Figure 11.) Their equations are as follows:

$$K_\circ\,(D^\circ \in AD): \quad \left(\frac{v}{v_D} - \frac{w}{w_D}\right)\left(\frac{u^2}{u_E u_F} + \frac{vw}{v_D w_D} - \frac{wu}{w_D u_D} - \frac{uv}{u_D v_D}\right) = 0$$

$$K_\circ\,(D^\circ \in BD): \quad \left(\frac{w}{w_D} - \frac{u}{u_D}\right)\left(\frac{v^2}{v_E v_F} - \frac{vw}{v_D w_D} + \frac{wu}{w_D u_D} - \frac{uv}{u_D v_D}\right) = 0$$

$$K_\circ\,(D^\circ \in CD): \quad \left(\frac{u}{u_D} - \frac{v}{v_D}\right)\left(\frac{w^2}{w_E w_F} - \frac{vw}{v_D w_D} - \frac{wu}{w_D u_D} + \frac{uv}{u_D v_D}\right) = 0$$

---

[29] This result is by far the most difficult in this note to verify algebraically. One approach is to encode the condition that A and its isotomic conjugate relative to $\triangle PBC$ are collinear with $\triangle PBC$'s "D^-" (that is, its X_{3260}); likewise with B and $\triangle APC$. Eliminating P's w-coordinate from the system leaves an 17000-term barycentric polynomial in u and v that factors into expressions corresponding to solutions D, E, F; a final, 1600-term factor lacks the required symbolic symmetry to yield a triangle center, and so is extraneous. A more-direct demonstration is desirable.

[30] In contrast to the previous result, this one —when formulated in terms of *tripolar coordinates*— is an almost-trivial consequence of the fact that each point P on K_+ is such that Euler lines (and Brocard axes) of $\triangle PBC$, $\triangle APC$, $\triangle ABP$, and $\triangle ABC$ concur. See [5], Corollary 6, wherein our K_+ corresponds to their $\mathcal{C}_2$.

[31] See the *ETC* Glossary "Ceva conjugate" entry. The Q-Ceva-conjugate of P has barycentric coordinates

$$u_P\left(-\frac{u_P}{u_Q} + \frac{v_P}{v_Q} + \frac{w_P}{w_Q}\right) \;:\; v_P\left(\frac{u_P}{u_Q} - \frac{v_P}{v_Q} + \frac{w_P}{w_Q}\right) \;:\; w_P\left(\frac{u_P}{u_Q} + \frac{v_P}{v_Q} - \frac{w_P}{w_Q}\right)$$

The D^+-Ceva-conjugate of D is Kimberling's X_{2132}; the D^- counterpart is not currently listed in *ETC*.

Observe that the conic component of each equation has no explicit reference to conjugate elements. We can in fact describe the conic as the one passing through, for instance, $BCEF$ and having tangents at B and C concur at D; it *happens to* also include E° and F°.

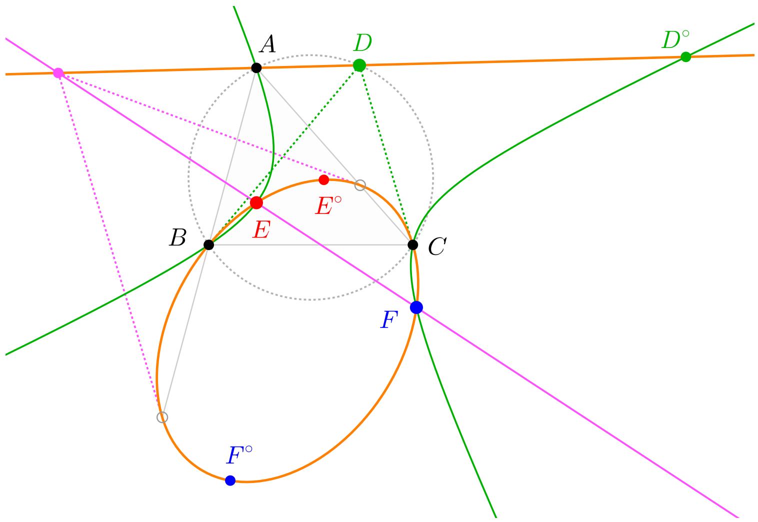

FIGURE 11. Circumcubic $K_\circ$ for $D^\circ \in AD$, the union of line AD and conic $BCEE^\circ FF^\circ$

Circumcubics Ω_{-P}, Ω_{+P}, $\Omega_{\circ P}$. Circumconic Γ_D contains D^-, E, F; its isotomic conjugate line Γ_D^- contains D, E^-, F^-. Therefore, the *union* of these sets is a (reducible) *circumcubic* containing the triad points and their isotomic conjugates. This is, of course, true for each triad member. We can express the unions, and their equations, thusly:

$$\Omega_{-P} \;:=\; \Gamma_P \cup \Gamma^-_{P_\star} \qquad \to \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_{P_\star}} vw \;\cdot\; \sum_{\text{cyc}} \frac{a^2\delta_a}{u_P} u \;=\; 0$$

As a result, all three circumcubics belong to a *pencil* on the nine points A, B, C, D, E, F, D^-, E^-, F^-. This pencil also contains K_-. Any member of such a pencil is expressible as a linear combination of any pair of members.[32] The triad cubics, and K_-, admit particularly-nice decompositions:

$$\Omega_{-D} \;\propto\; \Omega_{-E} + \Omega_{-F} \qquad\qquad K_- \;\propto\; \Omega_{-E} - \Omega_{-F}$$

As one might expect, an identical analysis goes through for cubics cobbled-together from our circumconics and their *isogonal* conjugate lines.

$$\Omega_{+P} \;:=\; \Gamma_P \cup \Gamma^+_{P_\star} \qquad \to \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_{P_\star}} vw \;\cdot\; \sum_{\text{cyc}} \frac{\delta_a}{u_P} u \;=\; 0$$

The pencil on A, B, C, D, E, F, D^+, E^+, F^+ contains Ω_{+D}, Ω_{+E}, Ω_{+F}, and K_+, and we have

$$\Omega_{+D} \;\propto\; \Omega_{+E} + \Omega_{+F} \qquad\qquad K_+ \;\propto\; \Omega_{+E} - \Omega_{+F}$$

And, generally, for isoconjugation P° with respect to Q on line X_1X_{31}, we can define

$$\Omega_{\circ P} \;:=\; \Gamma_P \cup \Gamma^\circ_{P_\star} \qquad \to \qquad \sum_{\text{cyc}} \frac{a^2\delta_a}{u_{P_\star}} vw \;\cdot\; \sum_{\text{cyc}} \frac{a^2\delta_a}{u_P u_D u_D^\circ} u \;=\; 0$$

[32]That is, if three members of the pencil have the equations $p(u,v,w) = 0$, $q(u,v,w) = 0$, $r(u,v,w) = 0$, then $r(u,v,w) \propto \alpha\, p(u,v,w) + \beta\, q(u,v,w)$ for some α and β (and constant of proportionality) that are symmetric expressions in side-lengths a, b, c. In the decompositions presented, we abuse notation so that the name of a cubic also represents the left-hand side of its equation.

which gives rise to corresponding decompositions

$$\Omega_{\circ D} \;\propto\; \Omega_{\circ E} + \Omega_{\circ F} \qquad\qquad K_\circ \;\propto\; \Omega_{\circ E} - \Omega_{\circ F}$$

Circumcubic $K_\varnothing$. A final example of a (reducible) circumcubic through the triad members is obvious but not-quite-trivial: define $K_\varnothing$ as the union of line EF and the circumcircle of $\triangle ABC$.

$$K_\varnothing : \sum_{\text{cyc}} \frac{a^2\delta_a}{u_D u_E u_F} u \cdot \sum_{\text{cyc}} \frac{a^2}{u} = 0$$

Tangents to the curve at A, B, C do not concur, so $K_\varnothing$ is not an instance of $K_\circ$.[33] Nevertheless, the cubic *is* self-isoconjugate, here with respect to Q (not on line X_1X_{31} nor in *ETC*) given by:

$$Q := \frac{a^3\delta_a}{u_D u_E u_F} : \frac{b^3\delta_b}{v_D v_E v_F} : \frac{c^3\delta_c}{w_D w_E w_F}$$

In particular, the cubic contains the Q-isoconjugates of the triad:

$$D^\varnothing := \frac{u_E u_F}{\delta_a} : \cdots : \cdots \qquad E^\varnothing := \frac{u_F u_D}{\delta_a} : \cdots : \cdots \qquad F^\varnothing := \frac{u_D u_E}{\delta_a} : \cdots : \cdots = X_{1304}$$

Some facts about these points:

- Point $D^\varnothing$ lies on the line EF. It is the non-vertex point common to circumcubics Φ_E and Φ_F introduced (along with points M_E and M_F) at the end of Section 4. For $P = D$ or E, tangents to Φ_P at $D^\varnothing$ and M_P meet at $P^\varnothing_\star$. (Figure 12.)

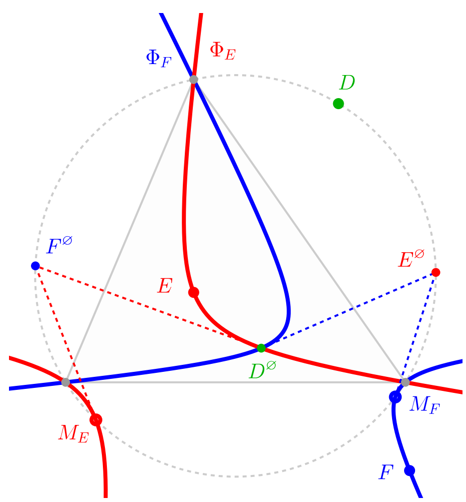

Figure 12. Circumconics Φ_E and Φ_F again

- $E^\varnothing$ and $F^\varnothing$ lie on the circumcircle, and their constructions via concurrent lines are explained under "Reflections and perspectors" in Section 7.

- Lines $EF^\varnothing$ and $FE^\varnothing$ meet at X_{477}, the reflection of D in the Euler line (also, the fourth point where the Yiu hyperbola, Γ_D, meets the circumcircle). The tangent to $K_\varnothing$ at D meets EF at the X_{477}-Ceva-conjugate of D.

[33]The $\varnothing$ mark was chosen to evoke the fusion of line and circle. The visual assertion of "not $K_\circ$" is a nice bonus.

- Line $DE^{\varnothing}$ is tangent at D to circumconics Γ_E, and (by footnote 12) it contains the point where EF meets Γ_F. Likewise, $DF^{\varnothing}$ is tangent to Γ_F and contains the point where EF meets Γ_E. (Figure 13, left.) The meeting points (neither of which is in ETC) have these coordinates

$$EF \cap \Gamma_F = \frac{u_D u_F}{u_E} : \cdots : \cdots = \frac{DF}{E} \qquad EF \cap \Gamma_E = \frac{u_D u_E}{u_F} : \cdots : \cdots = \frac{DE}{F}$$

- Circles $\bigcirc D^{\varnothing}EF$ (that is, line EF), $\bigcirc DE^{\varnothing}F$, $\bigcirc DEF^{\varnothing}$ concur at a point X_{2132}, the D^+-Ceva-conjugate of D; correspondingly, $\bigcirc DE^{\varnothing}F^{\varnothing}$ (the circumcircle), $\bigcirc D^{\varnothing}EF^{\varnothing}$, $\bigcirc D^{\varnothing}E^{\varnothing}F$ concur at the Q-isoconjugate of X_{2132}. (Figure 13, right.)

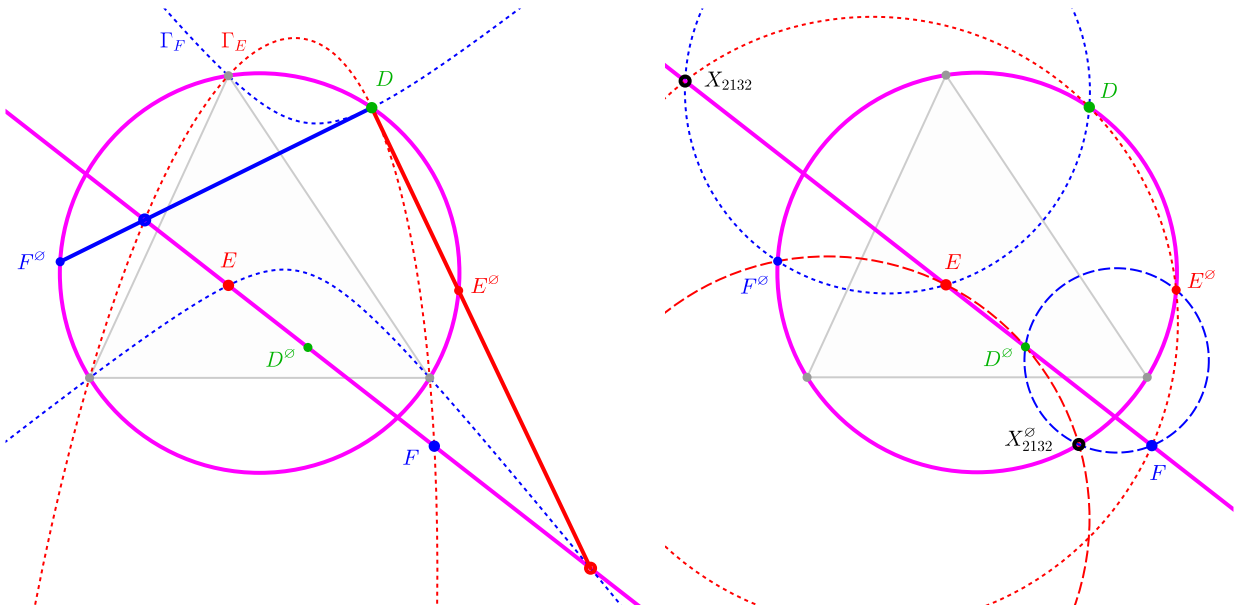

FIGURE 13. Circumcubic $K_{\varnothing}$, the union of the circumcircle and line EF

6. Complex Combinations

Consider our $\triangle ABC$ embedded in the complex plane, and let P_n be the *offset* $P - X_n$ for triangle center X_n. We observe the following *elementary symmetric* offset relations:

$$A_n + B_n + C_n + D_n = 0 \qquad n = 6699$$
$$A_n + B_n + C_n + E_n = 0 \qquad n = 5$$
$$A_n + B_n + C_n + F_n = 0 \qquad n = 45694$$

$$A_nB_n + A_nC_n + A_nD_n + B_nC_n + B_nD_n + C_nD_n = 0 \qquad n \in \{3, 125\}$$

$$A_nB_nC_n + A_nB_nF_n + A_nC_nF_n + B_nC_nF_n = 0 \qquad n \in \{1511, \bullet, \bullet\}$$

The three linear relations effectively define $G_D := X_{6699}$, $G_E := X_5$ (the nine-point center), and $G_F := X_{45694}$, such that each G_P is the vertex centroid of $ABCP$; geometrically, G_P is the dilation of P in $\triangle ABC$'s centroid $G := X_2$ by scale factor $1/4$. (Figure 14.) Each such G_P is a shared triangle center for triangles $\triangle ABC$, $\triangle PBC$, $\triangle APC$, $\triangle ABP$.

The quadratic relation highlights the fact that the four triangles defined by $P = D$ actually share a common circumcircle and Jerabek hyperbola (referred to as Γ_F earlier in this note), hence

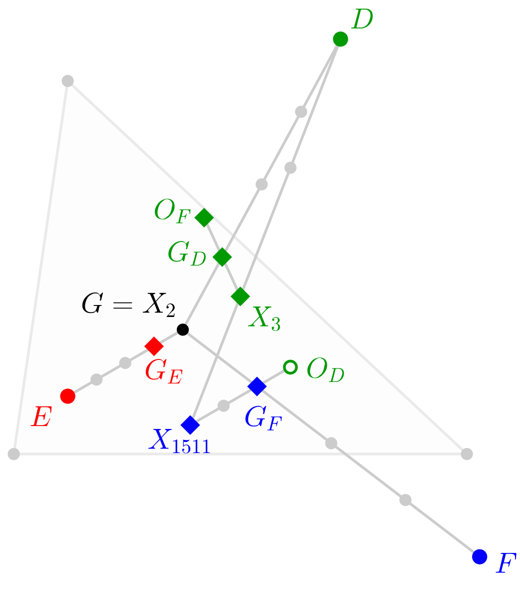

FIGURE 14

have a common circumcenter X_3 and hyperbola center $O_F = X_{125}$. The midpoint of these shared points is G_D. That the shared points lead to the uncomplicated offset relation seems non-obvious.

The cubic relation identifies X_{1511} (known as the *Fermat crosssum*; also, the midpoint of DF^+, the dilation of O_D in G_F by scale factor -2, and the dilation of D in circumcenter X_3 by scale factor $-1/2$) as a triangle center common to the triangles defined by $P = F$. Two other triangle centers (not indexed in *ETC*) satisfy the cubic offset relation, but their barycentric representations are a bit ugly; these centers are mutual reflections in O_D, so that the triangle they form with X_{1511} has centroid G_F.

Solving the counterpart quadratic and cubic offset relations involving other triad members yields (non-indexed) triangle centers with complicated coordinates. On the other hand, it's perhaps worth noting that the triangles defined by $P = F$ share rectangular hyperbola Γ_D through $ABCF$, and therefore center $O_D = X_{3258}$; this point gives the non-elementary cubic offset relation

$$\begin{aligned}3(A_n + B_n + C_n + F_n)(A_nB_n + A_nC_n + A_nF_n + B_nC_n + B_nF_n + C_nF_n)\\ = 2\,(A_nB_nC_n + A_nB_nF_n + A_nC_nF_n + B_nC_nF_n) \qquad n = 3258\end{aligned}$$

7. MISCELLANEA

Functionalities. If we treat X_n as a point-valued function on a triangle's vertices, then the defining property of a tetradic center of $\triangle ABC$ is

$$X_n(X_n(A, B, C), B, C) = A, \quad X_n(A, X_n(A, B, C), C) = B, \quad X_n(A, B, X_n(A, B, C)) = C$$

which, as we know, is satisfied by $n = 4$, 74, and 1138 (at least).

Matt F. [7] has observed the following about $E = X_4$, given auxiliary point O:

$$X_4(A, B', C') = X_4(A', B, C') = X_4(A', B', C) \qquad =: M \tag{19}$$

where

$$A' := X_4(O,B,C) \qquad B' := X_4(A,O,C) \qquad C' := X_4(A,B,O)$$

The common derived point has barycentric coordinates

$$M \;=\; \frac{1}{\left(-\frac{u_O}{u_E}+\frac{v_O}{v_E}+\frac{w_O}{w_E}\right)+2u_O^+v_Ow_O} : \cdots : \cdots$$

Replacing X_4 with X_{74} or X_{1138} in (19) doesn't work. This author is currently unaware of counterpart relations for tetradic centers D and F.

Relatedly, $\bigcirc AB'C'$, $\bigcirc A'BC'$, $\bigcirc A'B'C$ are concurrent for $n = 4$ and $n = 74$, but not $n = 1138$. For $n = 4$, the point of concurrency has coordinates that curiously incorporate those of M above

$$\frac{1}{u_M^-\left(u_O^+ + v_O^+ + w_O^+\right) - u_O^-(u_O+v_O+w_O)\left(u_E^+ + v_E^+ + w_E^+\right)} : \cdots : \cdots$$

For $n = 74$, the coordinates are exceedingly unwieldy. Note that, because X_{74} lies on the circumcircle of its host triangle, the point of concurrency *would be* the direct analog of M for $n = 74$, if such a point existed.

Reflections and perspectors. In the following, P_A, P_B, P_C are the reflections of point P in the $\triangle ABC$'s side-lines BC, CA, AB. Also, calling points X, Y, Z "perspective" means that "$\triangle XYZ$ is perspective with $\triangle ABC$"; that is, XA, YB, ZC are concurrent at point called the *perspector* of the two triangles.

- The locus of points P whose reflections P_A, P_B, P_C are perspective is known to be K_+.[34] (So, the triad members are among these points.) For any such point P and corresponding perspector P', the line PP' is parallel to the Euler line.
- Paul Yiu [18] observed that D_A, D_B, D_C are collinear and perspective (and asserted that D is the only point whose reflections have both properties); specifically, their line is perpendicular to the Euler line at E. The perspector is $D' := X_{5627}$.
- Points E_A, E_B, E_C lie on the circumcircle, and their perspector is $E' := E$.
- Points F_A, F_B, F_C have perspector $F' := X_{14451}$.
- For (P,Q,R) some permutation of the triad, the circumcenters of $\bigcirc P_AP_BP_C$ and $\bigcirc Q_AQ_BQ_C$ are collinear with $R_\star$.[35] (Figure 15.) This is simply a repackaging of some concurrence results from Section 3, since, for any point P, the circumcenter of $\bigcirc P_AP_BP_C$ is P^+, the isogonal conjugate of P.

  Also, $\bigcirc D_AD_BD_C$ contains E, and $\bigcirc E_AE_BE_C$ contains D, but $\bigcirc F_AF_BF_C$ contains no triad member.

Some related facts

- Lines E_AF_A, E_BF_B, E_CF_C concur at a point U (not indexed in *ETC*), and lines D_AE_A, D_BE_B, D_CE_C concur at $W = X_{1304}$.

  Lines F_AD_A, F_BD_B, F_CD_C do not concur. However, they determine a triangle perspective with $\triangle ABC$, with perspector V (not in *ETC*).

[34] See Gibert [9], "K001".

[35] Of course, we must make appropriate accommodation for the fact that "circle" $\bigcirc D_AD_BD_C$ is a line and its "circumcenter" D^+ is the Euler line's point at infinity.

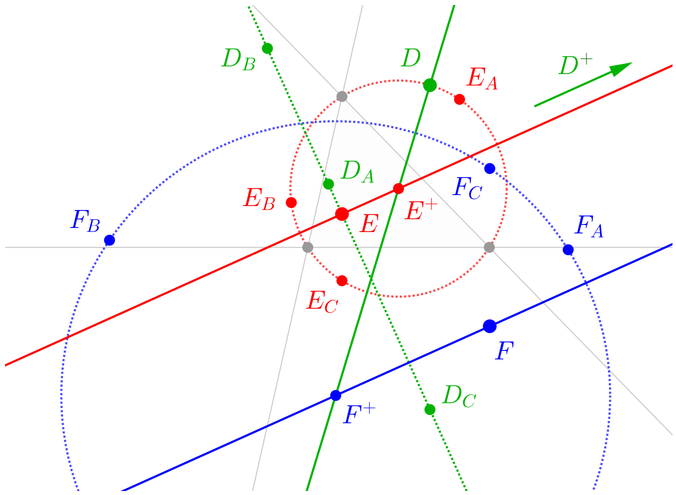

Figure 15

Points U, V, W lie on the circumcircle. (Figure **??**.) Their coordinates are as follows:[36]

$$U = \frac{u_D u_F}{\delta_a} : \cdots : \cdots \qquad W = \frac{u_D u_E}{\delta_a} : \cdots : \cdots \qquad V = \frac{u_D u_F}{\delta_a \left(6\tau^2 \delta_b \delta_c u_F - a^2 b^2 c^2\right)} : \cdots : \cdots$$

- Generally, $\bigcirc P_A BC$, $\bigcirc P_B CA$, $\bigcirc P_C AB$, $\bigcirc P_A P_B P_C$ concur at a point with coordinates

$$\frac{u_P}{\left(-\frac{u_P}{u_E} + \frac{v_P}{v_E} + \frac{w_P}{w_E}\right)(u_P + v_P + w_P) - \left(\frac{u_P^2}{u_E} + \frac{v_P^2}{v_E} + \frac{w_P^2}{w_E}\right)} : \cdots : \cdots$$

  For $P = D$, this point is $E = E'$. For $P = F$, the point is D'. For $P = E$, the expression for the point becomes undefined (understandably, because the four circles coincide with the circumcircle); however, as P *approaches* E along K_+, the point of concurrency approaches X_{1141}, the *Gibert Point*.

- The Yiu hyperbola (Γ_D) contains D' and $E' = E$ (the latter by definition), but not F'. Conic Γ_F contains $E' = E$ by definition, but neither D' nor F'; conic Γ_E contains none of D', E', F'.

A Search for Centers. A note about tetradic centers should probably include a few words about the search for other such centers. These are those words ...

Triad members D, E, F are in a class of triangle centers whose barycentric coordinates are homogeneous in the *squares* of side-lengths of $\triangle ABC$; this is convenient in that it avoids messy invocations of the square root. We can write the coordinates of such a center, T, thusly:

$$T \;:=\; f_2\left(a^2, b^2, c^2\right) : f_2\left(b^2, c^2, a^2\right) : f_2\left(c^2, a^2, b^2\right)$$

for some homogeneous rational function such that $f_2(x, y, z) = f_2(x, z, y)$; we may take f_2 to have degree -1. Applying the function to $\triangle TBC$, the resulting T-center will have these barycentric coordinates (relative to $\triangle ABC$):

[36]That's not a typographical error. While points V and W have coordinates derived from those of the points that determine them, point U —determined by E and F— has coordinates derived from those of D and F. Appealing to triadic duality doesn't seem to make sense of this situation.

$$\begin{aligned} T' := \quad & u_T f_2\left(a^2, \bar{b}^2, \bar{c}^2\right) \\ & : v_T f_2\left(a^2, \bar{b}^2, \bar{c}^2\right) + (u_T + v_T + w_T) f_2\left(\bar{b}^2, \bar{c}^2, a^2\right) \\ & : w_T f_2\left(a^2, \bar{b}^2, \bar{c}^2\right) + (u_T + v_T + w_T) f_2\left(\bar{c}^2, a^2, \bar{b}^2\right) \end{aligned}$$

where we have

$$\bar{b}^2 := |TC|^2 = \frac{a^2 v_T^2 + b^2 u_T^2 + (a^2 + b^2 - c^2) u_T v_T}{(u_T + v_T + w_T)^2} \qquad \bar{c}^2 := |TB|^2 = \frac{a^2 w_T^2 + c^2 u_T^2 + (a^2 - b^2 + c^2) u_T w_T}{(u_T + v_T + w_T)^2}$$

To make T tetradic, "all we need to do" is choose f_2 so that the last two barycentric coordinates of T' are identically zero for all a, b, c (and the first component is non-zero).[37] In slightly-more-compact form, we seek solutions to this functional system:

$$-\frac{f_2\left(a^2, \bar{b}^2, \bar{c}^2\right)}{f_2\left(a^2, b^2, c^2\right) + f_2\left(b^2, c^2, a^2\right) + f_2\left(a^2, b^2, c^2\right)} = \frac{f_2\left(\bar{b}^2, \bar{c}^2, a^2\right)}{f_2\left(b^2, c^2, a^2\right)} = \frac{f_2\left(\bar{c}^2, a^2, \bar{b}^2\right)}{f_2\left(c^2, a^2, b^2\right)}$$

A brute-force search for such solutions quickly gets out of hand.

- If the numerator of f_2 has degree 0, we can determine without too much difficulty that the function must have this form (up to a multiplied constant):

  $$f_2(x, y, z) = \frac{1}{x - (y + z)}$$

  Therefore, triad member E is the only tetradic center for this case.

- If the numerator of f_2 has degree 1, then already the situation explodes in symbolic complexity. Consider f_2 in the form

  $$f_2(x, y, z) = \frac{h_1 x + h_2(y + z)}{k_1 x^2 + k_2 x(y + z) + k_3 (y + z)^2 + k_4 yz}$$

  Expanding an ostensibly-vanishing coordinate of T' using this function yields a polynomial with over *three million* terms; whether it factors is beyond this author's available computing power to determine. Gathering terms by $a^p b^q c^r$ —each coefficient of which must vanish— yields just-over 700 equations in the h_i and k_i. *Assuming*[38] this author has correctly performed the appropriate casework, the equations reveal no new tetradic centers; rather, they merely reconfirm the tetradic nature of triad members D and (via simplification of f_2) E, as shown:

$(\ h_1, h_2\ ;\ k_1, k_2, k_3, k_4\ )$	$f_2(x, y, z)$	T
$(\ 1, 0\ ;\ 2, -1, -1, 4\ )$	$\dfrac{x}{2x^2 - x(y+z) - (y-z)^2}$	D
$(\ h_1, h_2\ ;\ h_1, h_2 - h_1, -h_2, 0\ )$	$\dfrac{1}{x - (y+z)}$	E

  The extraneous solutions have these forms[39]

[37] We can therefore ignore instances of T on the line at infinity, since such points have $u_T + v_T + w_T = 0$, causing all coordinates of T' to vanish simultaneously.

[38] The reader is advised to heed the wisdom of Felix Unger in this regard.

[39] For X_{524}, we can ignore the denominator of f_2 since it is symmetric in x, y, z.

$(h_1, h_2 \; ; \; k_1, k_2, k_3, k_4)$	$f_2(x, y, z)$	T
$(2, -1 \; ; \; 1, -2, 1, -4)$	$\dfrac{2x - (y + z)}{x^2 + y^2 + z^2 - 2xy - 2yz - 2zx}$	X_{524}
$(h_1, h_2 \; ; \; 1, -1, 0, 1)$	$\dfrac{h_1 x + h_2(y + z)}{(x - y)(x - z)}$	–

- If the numerator of f_2 has degree 3 or more, then brute-force symbolic manipulation seems impractical.

Acknowledgements

The author greatly appreciates Clark Kimberling's *Encyclopedia of Triangle Centers*, whose rabbit-holes of cross-reference revealed countless relations among points mentioned in this note. Likewise, Bernard Gibert's *Cubics in the Triangle Plane* catalogue and associated notes provided many key insights. Finally, the *GeoGebra* dynamic geometry application —with convenient incorporation of Kimberling centers and Gibert cubics— facilitated exploration and discovery of numerous unexpected connections, confirmed symbolically via Wolfram's *Mathematica*.

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